NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
4 Washington Place, New York 3, N. Y.

AMG-NYU No. 151

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
~~25 Waverly Place~~, New York 3, N. Y.

A GAS DYNAMICAL FORMULATION FOR WAVES AND COMBUSTION IN PULSE-JETS

Prepared under
Navy Contract NOa(s)-7370
By the
Institute for Mathematics and Mechanics
New York University
(Formerly The Applied Mathematics Group)

June 1946

NYU
AMG 151 c.1

PREFACE

In pulse-jets of the type used to propel the German Buzz Bomb, turbulence appears to have considerable influence on combustion and on effective flame speeds and therefore on the oscillation and flow of the gas through the motor. The following report assumes that these effects can be accounted for mathematically by use of parameters (to be determined by controlled fundamental experiments) in general one-dimensionalized thermodynamical gas equations. Idealized numerical cases are considered. It is intended that these equations be used to investigate optimum internal conditions which, it is hoped, will lead to specific recommendations for the design of more effective pulse-jets, as well as to a fundamental understanding of automatic reignition and other characteristic phenomena. Previous attempts at a theory of intermittent jet motors have ignored flame or wave propagation as well as turbulence effects.

Professor J. K. L. MacDonald, the author of the present report, was greatly assisted by Dr. S. A. Schaaf in particular and by other members of the staff of the Institute for Mathematics and Mechanics in New York University. This important investigation has been carried out in connection with a research program under Contract No. NOa(s)-7370 with the Bureau of Aeronautics of the Navy Department.

R. Courant
Director
Institute for Mathematics and Mechanics

DEFINITION OF SYMBOLS

	Page		Page
a	7	p	8
a_o	22	p_o	15
c_o	22	p_1	14
c_1	22	p_2	14
C_p	8	p_3	14
C_v	8	R	8
$\dot{E}$	8	$S(x)$	6
ΔE	17	t	7
F	8	τ	8
f	17	τ_o	22
γ	8	τ_1	22
γ_o	22	θ	8
γ_1	22	θ_o	22
h	15	θ_1	22
I	23	u	8
K	19	u_o	15
L	6	U	8
L^*	14	x	7
m	8	z	15
m_o	23	Z	22
$\dot{m}$	23		

(Other symbols used in the Appendices are defined there.)

CONTENTS

Page

1. Introduction 1

2. Previous Theoretical Treatments of Waves in Pulse-Jets 3

3. One-dimensionalized Formulation for Pulse-jets 6

    3.1 Hydro-thermodynamical equations for the one-dimensionalized pulse-jet 9

    3.2 On approximations for use in solving the one-dimensionalized equations 12

        (i) Shock front conditions 12
        (ii) Intake and exhaust end conditions 14
        (iii) Combustion and flame speeds 16
        (iv) Perturbation series for the equations 19
        (v) Explicit treatment of an idealized case 21

4. Concerning the nature of desired experimental data on pulse-jets 28

Appendices

I Derivation of the equations in (m,t) 31

II Perturbation Series 33

III Explicit Solution of an Idealized Case 37

IV On Analytical and on Numerical Methods for Direct Treatment of the One-dimensionalized Thermo-hydrodynamical Non-Linear Equations 57

References 62

A GAS DYNAMICAL FORMULATION FOR WAVES AND COMBUSTION IN PULSE-JETS

SUMMARY

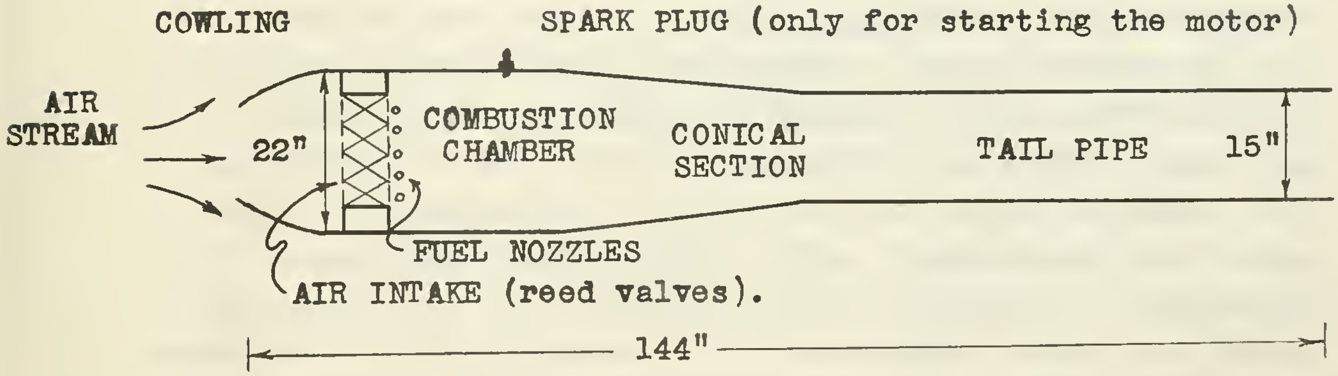
A brief outline is given of known experimental data on the motions of the gases and flames in pulse-jets (such as the German "Buzz-bomb" motor). The considerable assumptions made in previous theoretical treatments are indicated and a rather comprehensive one-dimensionalized hydro-thermodynamical formulation for pulse-jets is then developed. Terms are included corresponding in particular to: (i) wave and flame propagation throughout the jet, (ii) effective duration of burning in the turbulent gases, and (iii) non-uniform cross-section and boundary conditions at the intake and at the exhaust ends. An outline is given of the nature of further experimental data required for the explicit determination of boundary conditions and of the parameters involved in combustion and flame propagation in a turbulent mixture. It is strongly urged that such data be obtained in order to make detailed applications of the theory to practical questions of efficiency and design. An idealized example is solved explicitly (analytically to first order terms) on the basis of presumably reasonable assumptions concerning such data, and is seen to represent observed features and to contradict some assumptions made in previous theoretical treatments. In particular pressure effects due to the moving flame fronts appear to be important.

A variety of alternative forms is presented to provide a framework into which future observations and explicit examples may fit. It is hoped that the existence of such a framework will help to stimulate really fundamental experimental research on pulse-jets.

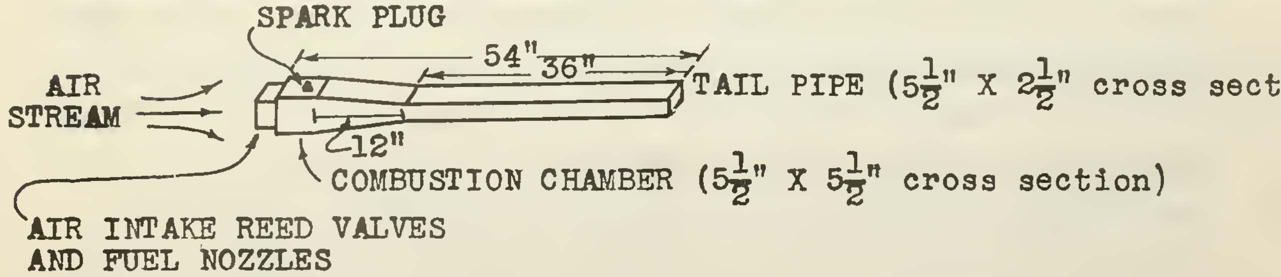
A GAS DYNAMICAL FORMULATION FOR WAVES AND COMBUSTION IN PULSE-JETS

1. Introduction

Following the development of the V.1 "Buzz Bomb" by the Germans, various groups in England and the U.S.A. started general investigations of pulse-jet motors. Among these groups were the following: The R.A.E. (Royal Aircraft Establishment, Farnborough, England),(0) the U.S. Army Air Force (at Wright



(a) German V.1 type motor with axial symmetry. Frequency about 50 cycles/se


SPARK PLUG
AIR STREAM
54"
36"
TAIL PIPE ($5\frac{1}{2}$" X $2\frac{1}{2}$" cross sect
12"
COMBUSTION CHAMBER ($5\frac{1}{2}$" X $5\frac{1}{2}$" cross section)
AIR INTAKE REED VALVES
AND FUEL NOZZLES

(b) Annapolis "Resojet" type motor. Frequency about 100 cycles/sec.

Fig. 1: Approximate structure and dimensions of standard pulse-jets.

Field),(1) the N.A.C.A. (National Advisory Council for Aeronautics, Cleveland),(2) a BuAer group (at Annapolis),(3) the Aerojet Corp. in California,(4) the Reaction Motors Corp. in New Jersey, the Jet Propelled Missiles Panel of the O.S.R.D.(5) (for the U.S. Navy) and the AMG-NYU (Applied Mathematics Group at New York University).

The operation of standard types of pulse jets, such as are represented in Figure 1, is roughly as follows: The motor is placed in an air stream which provides a "ram pressure" to force air through the reed valves (see Fig. 1) into a region where the air is mixed with a continually sprayed fuel (such as gasoline). The mixture is initially fired by means of a spark-plug, and the burning raises the gas pressure in the combustion chamber, ultimately causing the reed valves to close and exhaust gases to flow out through the tail pipe, a thrust being developed during the process. The exhaust generates rarefaction waves which soon reduce the pressure behind the reed valves until they open again and admit more air. Then, even without further use of the spark-plug a self firing occurs in the gas, apparently in a central region near the section through the spark plug, and a regular cycle is established. The mechanism of the self-firing is not clearly understood but it seems probable that pressure waves reflected from the ends of the motor cause a local heating and mixing of the gas sufficient for its ignition near the interface between the incoming (unburnt) mixture and the previously burnt gases. The flame apparently spreads from the region of re-ignition with a relatively slow speed, in a manner indicated in Figure 5; the process thus resembles a wave propagation rather than an instantaneous explosion. (In an ordinary internal combustion engine the distances involved in wave and flame propagation are relatively small and the state of the gas is effectively homogeneous at all times.)

Using fairly crude instruments most of the above mentioned groups of investigators have made measurements of some sorts of average values (over the cycle) of the following quantities: thrust, drag, ram pressure, rates of flow of fuel and air, and pressures and temperatures at various points. Some attention was devoted to the effects produced by changes in structures and in fuels. A few nominal pressure-time oscillograms were obtained but were subject to unknown perturbations due to violent thermal, mechanical and turbulent oscillations in and around the pick-ups. Preliminary studies of the motions of flames and of exhaust gases were made at Wright Field, at Cleveland and at Annapolis, by use of high speed movie cameras (2000 frames per second). Really fundamental and accurate experimental work remains to be done. Under contract to BuAer, groups at the N.R.L. (Naval Research Laboratory, Washington), at the Aerojet Corporation and at New York University hope to carry through fairly comprehensive programs of experimental work on the "internal ballistics" of pulse-jets. The results of such investigations would provide basic data for use in a theoretical treatment such as that which will be discussed in Section 3, and which, it is hoped, will lead to improvements in the designs of pulse-jets. The nature of desired experimental data is indicated in Section 4. It is strongly urged that such data be obtained by groups with the required facilities.

2. Previous Theoretical Treatments of Waves in Pulse-Jets.

From the few relevant reports received by the AMG-NYU it would appear that British observers first noted that the frequency of the V.1 motor under normal operating conditions is almost the same as that of the fundamental mode in an organ pipe having the same dimensions and the same (average) gas temperature as the motor.

In a Wright Field report(1 the V.1 motor is treated as a sort of Helmholtz resonator.(6 The usual Helmholtz resonator has roughly the shape of a flask, as indicated in Fig. 2, with the volume V and its cross-section considerably greater than those of V'. Then V acts like a spring against which oscillates the mass of V' (plus small effective masses at the ends of V'). In the Buzz Bomb motor V and V' are practically undistinguishable from one another and the assumptions used in the standard resonator theory are violated. It is most probably merely a numerical accident that the observed frequency of the standard Buzz Bomb is in close agreement with the theoretical resonator frequency, especially since the Wright Field report treats V + V' as the spring for V'. A considerably refined (but similar) treatment is given in two reports (5 by Professor Neil P. Bailey and his associates: The pulse-jet is again treated mathematically on the assumption that the state of the gas in the combustion chamber remains uniform as it varies with time and produces forced oscillations of acoustic type in the tail pipe. Some effects due to non-instantaneous burning of the fuel are included by assuming a periodic exponential pulse form for the quantity $\frac{1}{\theta}\frac{d\theta}{dt}$, where θ is the absolute temperature in the combustion chamber at time t. Rather comprehensive numerical calculations were made by Bailey but the assumption of uniformity of state (i.e. no wave motion) throughout the combustion chamber, as well as assumptions about the bulk modulus and the temperature of the gas, seem, on the basis of a more complete theory to have little justification.

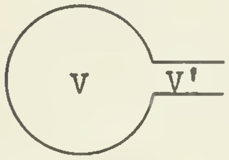


Fig. 2

In an informal memorandum(7 by Professor H. Lewy of the AMG-NYU there is indicated an interesting type of formulation for waves in pulse-jets. Although Lewy's formulation neglects the non-instantaneous character of the combustion and makes some assumptions of doubtful validity concerning the velocities and densities in the gases, it may lead to useful generalizations after more experimental data are available. The method depends upon the effective elimination from the (one dimensional) equations of motion and continuity, of the position variable x. This elimination results from an x-integration of both sides of the equations, coupled with the assumptions that $\int_0^L u(x,t)dx/u(L,t)$ and $\int_0^L \rho(x,t)dx/\rho(0,t)$ are both independent of the time t, and that the entropy of the gas remains constant. Here $u(x,t)$ is the gas velocity and $\rho(x,t)$ the gas density, and $x = 0$ and $x = L$ respectively correspond to the positions of the intake and the exhaust ends of the pulse-jet. Non-linear differential equations then arise for $u(L,t)$ and $\rho(0,t)$ and are solved (for special initial conditions) by a combination of analytical and numerical methods. Generalizations of Lewy's method, to include burning periods and justified relations between averages, are planned in future investigations when basic experimental data are available.

German articles covering their war research on theories of pulse-jets are now becoming available (see reference 15). All such articles thus far received have not considered flame motion and non-instantaneous combustion, but they have given extensive strictly numerical analyses of non-linear, isentropic wave-motion in open and semi-open tubes of variable cross-section.

3. One-dimensionalized Formulation for Pulse-jets.

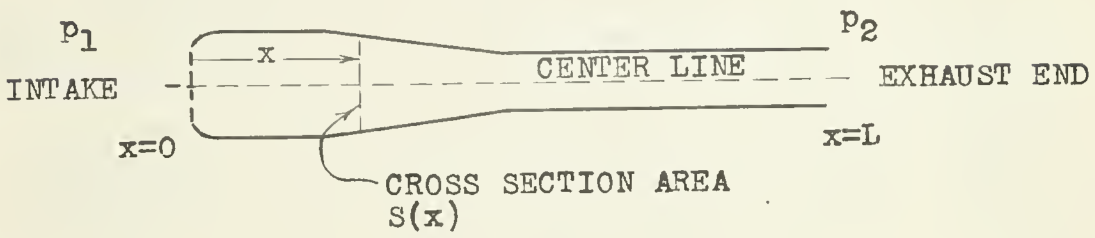


Fig. 3 Characteristic shape of the interior of pulse-jets.

p_1 = ram pressure. p_2 = atmospheric pressure outside the exhaust pipe.

The usual pulse-jet is a steel tube of mildly non-uniform cross-section (circular or rectangular) with the exhaust end (see Fig. 3) constantly open and with the size of the relatively small openings at the intake end varying with time in a way depending on the varying effective pressure difference between the two sides of the intake, and also depending on the constants of the reed valves.

Although detailed observations are still lacking it seems definite that in the pulse-jet there is considerable turbulence, probably traceable mainly to the reed valves and to inhomogeneity in the fuel-air mixture. Correspondingly the combustion appears to proceed inhomogeneously in cluster form rather than in the relatively slowly moving idealized plane sheet form to which the LeChatelier theory and its recent extensions apply.(8 In fact it is not unlikely that the flames spread in the form of turbulently distorted and interpenetrating thin shells which make entire regions of the gas glow with combustion for several milliseconds (see Fig. 5). Presumably the turbulence greatly increases the effective rate of diffusion of heat and of chain-carriers and thereby produces the observed high rates of spread of the flames. Locally the combustion of particles is probably almost instantaneous after their ignition by heat and chain-carriers from adjacent particles, but in a given layer of the swirling mixture of thickness Δx at x (see Fig. 3) only a small fraction of the molecules may undergo combustion during any small interval of time Δt. Correspondingly, locally, the concepts of thermodynamic state may not apply, but the concepts of average thermodynamic state and of effective non-zero duration of burning would be expected to have a meaning for the layer as a whole. The term "flame speed" could be applied to the effective rate of advance of combustion from layer to layer along the x direction. From this viewpoint it would seem quite probable that the cross-sectional averages of salient physical quantities can be closely approximated by use of a one-dimensionalized model (with refinements awaiting the development of adequate theoretical treatments of turbulence and combustion, as well as adequate experimental studies). In such a one-dimensionalized model the state of the gas is taken, in effect, as uniform over the cross-section S(x) (see Fig. 3), and the following symbols may be introduced (typical consistent units are indicated for use in subsequent equations): At distance x(cm) from the intake at time t(sec) let

a = Lagrange coordinate, i.e. the value of x at t = 0 for the particle (layer) which is at x at time t (a therefore varies from particle to particle).

m = 'pseudo-Lagrange coordinate' with the following meaning: For the particle (layer) which reaches the position x at time t, m denotes the total mass of gas that has flowed through the intake from some convenient reference time t_0 until the time of arrival $t(m)$ of that particular particle at the (inside end of) the intake. Each particle therefore has its own value for m.

u = velocity (cm/sec) of the gas relative to the tube.

F = velocity (cm/sec) of flame front (at x) relative to the tube.

p = pressure (dynes/cm^2static) of the gas.

ρ = density (gm/cm^3) of the gas.

$\tau = 1/\rho$ = specific volume (cm^3/gm).

θ = absolute temperature ($^{\circ}$Kelvin) relative to the moving gas.

$\dot{E}$ = effective rate of release of heat energy per unit mass of gas mixture (ergs/gm sec) (at x at time t), that is the sum of the rate of release of chemical energy and of the rate of inflow (per unit mass) of heat by conduction and convection from adjacent regions. The values of $\dot{E}$ should be determined from experiments such as are indicated in section 4.

U = internal energy (ergs/gm) of the gas.

C_v and C_p = specific heats (energies)(ergs/gm$^{\circ}$K) at constant volume and constant pressure respectively.

$\gamma = C_p/C_v$ = ratio of specific heats.

$R = C_p - C_v$ = "ideal gas constant" (ergs/gm$^{\circ}$K).

3.1 Hydro-thermodynamical equations for the one-dimensionalized pulse-jet.

Note that the above definition for R does not refer to the customary 'mole' unit but is inversely proportional to the average molecular weight of the mixture at the position and time concerned. For commonly used fuel-air mixtures R remains nearly constant but $\gamma-1$ drops from about .4 to .2 as θ rises from 0° to about 2500°C during the combustion. Moreover, for such mixtures the equation of state is commonly held(9 to be represented fairly accurately by

$$(1) \qquad p\tau = R\theta \quad \text{(R constant)},$$

and the change in internal energy by

$$(2) \qquad dU = C_v d\theta = \frac{d(p\tau)}{\gamma-1}$$

In ordinary gas dynamics without combustion, the adiabatic relation

$$(3) \qquad -pd\tau = C_v d\theta = \frac{d(p\tau)}{\gamma-1} \quad \text{(whence } p\,\tau^{\gamma} = \text{const. for } \gamma \text{ const.)}$$

is usually employed, and it can be derived from the first law of thermodynamics and the relations (1) and (2). In the present formulation the symbol $\dot{E}$ is introduced to represent all heat contributions. (1), (2) and the first law of thermodynamics then yield the following analogue of the adiabatic relation (3):

$$(4) \qquad \dot{E}dt - pd\tau = C_v d\theta = \frac{d(p\tau)}{\gamma-1}$$

Here the differentials refer to the change in the thermodynamic state of a given moving particle (layer) during the time interval dt. For the case $\dot{E} = 0$ (4) reduces to (3). An enthalpy equation can be obtained by combining (4) with an equation of motion such as is given below, or alternatively by considering the entire flux of energy into a given particle (layer). As will be explained in Section 3.2 the introduction of shock discontinuity conditions is apparently an unnecessary refinement.

There are various pairs of independent variables available for specifications in pulse-jets. Eulerian coordinates (x,t)(10 are probably most convenient for discussing steady flows, but in non-steady flames they lead, in general, to rather cumbersome relations. Lagrangian coordinates (a,t)(10 keep track of each particle and are especially convenient for treating waves and combustion in initially resting gases in the tube; but inflows lead to some complications which can be avoided by use of pseudo-Lagrangian coordinates (m,t) (where m is as defined on page 8). The complications concerned are due to the fact that the Lagrange coordinate a of a particle which enters the intake after t = 0, depends sensitively upon the dynamics of the flow outside the motor.

In Eulerian coordinates (x,t)(10

$$(5) \qquad S(x)\rho_t(x,t) + [S(x)\rho(x,t)\,u(x,t)]_x = 0$$

is the equation of continuity, and

$$(6) \begin{cases} S(x)[\rho(x,t)\,u(x,t)]_t + [S(x)\,\rho(x,t)\,u^2(x,t)]_x + S(x)\,p_x(x,t) = 0 \\ \text{or, alternatively} \\ u_t(x,t) + u(x,t)\,u_x(x,t) + p_x(x,t)/\rho(x,t) = 0 \end{cases}$$

is the equation of motion. Here subscripts x and t represent corresponding partial derivatives and S(x) is as defined in Fig. 3 (page 6). By interpreting

$$\frac{d}{dt} \text{ as } \frac{\partial}{\partial t} + u(x,t)\frac{\partial}{\partial x} ,$$

equation (4) can be expressed in terms of Eulerian coordinates. Thus:

$$(7) \qquad (\gamma-1)\dot{E} = \gamma p\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\tau + \tau\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p$$

In Lagrangian coordinates (a,t) with $x(a,t) = a$ at $t = 0$ the hydro-thermodynamical equations (4), (5) and (6) become respectively

$$(8) \quad \dot{E} - p(a,t)\,\tau_t(a,t) = C_v\,\theta_t(a,t) = [p(a,t)\,\tau(a,t)]_t/(\gamma-1),$$

$$(9) \qquad x_a(a,t) = \frac{S(a)\,\tau(a,t)}{S(x)\,\tau(a,0)}$$

and

$$(10) \qquad x_{tt} = -S(x)\,\tau(a,0)\,p_a(a,t)/S(a).$$

In pseudo-Lagrangian coordinates (m,t) the corresponding equations (see Appendix I) are:

$$\dot{E} - p(m,t)\,\tau_t(m,t) = C_v\,\theta_t(m,t) = [p(m,t)\,\tau(m,t)]_t/(\gamma-1),$$

(11) whence

$$(\gamma-1)\dot{E} = \gamma p\tau_t + \tau p_t \;;$$

$$x_m(m,t) = -\tau(m,t)/S(x) \qquad (12)$$

and

$$x_{tt}(m,t) = S(x)\, p_m(m,t) \qquad (13)$$

where m denotes the mass of the mixture that has flowed through the intake from some reference time, t_o, until the time t(m) of entry there of the particles which (in the one dimensionalized model) will reach x at time t. Symbolically

$$m = \int_{t_o}^{t(m)} [S(0+)\, x_t/\tau]\, dt_{\,x=0+}$$

where S(0+) is the cross-section just following the valves at x = 0. At the intake end the mass flux is given by

$$S(x)\, x_t(m,t)/\tau(m,t) \rightarrow \frac{dm}{dt(m)} \text{ for } x(m,t) \rightarrow 0+ \ , \qquad (14)$$

with $\frac{dm}{dt(m)}$ of course zero during intervals when the reed valves are shut.

3.2 On approximations for use in solving the one-dimensionalized equations.

(i) Shock front conditions. It is well known(11 that compression waves (as well as explosions) can give rise to shock fronts across which entropy and other physical quantities take jumps in values. It might therefore be feared that a treatment of pulse-jets which implicitly ignores the possibility of shock waves would be of rather limited value. Such a fear appears to be groundless when the pressure oscillations and the burning rates are as small as rough measurements in standard pulse-jets indicate. Under such circumstances any shocks that may arise in the gas mixture would be expected to be so weak that p,u and τ in the one-dimensionalized model could be regarded as continuous or as limits of continuous approximations. In fact if one compares pressure distributions calculated respectively on the basis of (linear) acoustic wave and (non-linear) shock wave equations, for a problem such as that indicated in Figure 4, one finds only a few percent difference between the two.

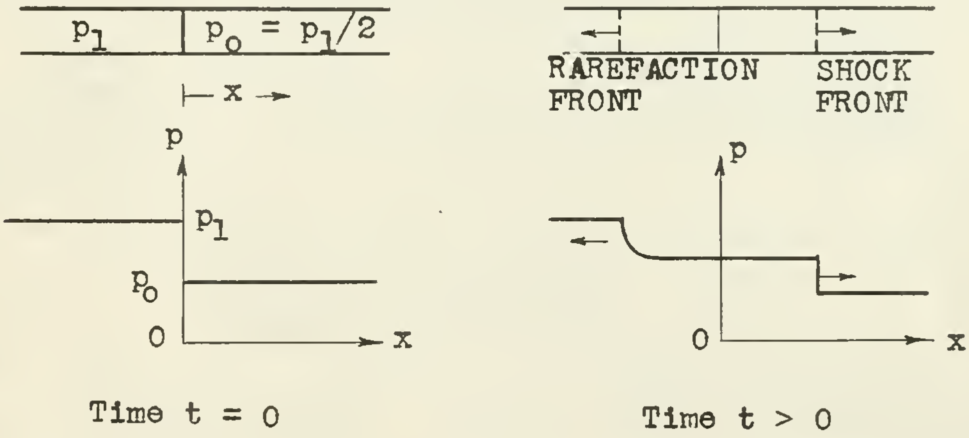


Fig. 4 Waves produced in a tube in which there is an initial discontinuity in the pressure p of the gas.

The omission of shock conditions permits the development of relatively simple iteration schemes for solving the equations analytically rather than purely numerically, a feature which is desirable from a practical viewpoint because of the many parameters whose influence must be estimated for design and other purposes.

(ii) Intake and exhaust end conditions. At the exhaust end ($x = L$, see Fig. 3), rough pressure measurements coupled with notions used in acoustic theory(12 suggest that to a good approximation

(15) $$p = p_2 \quad \text{at} \quad x = L^*$$

where p_2 is the effective atmospheric pressure outside the exhaust pipe and where $L^* - L$ is an "open end correction"(12 which is of the order of half the diameter of the exhaust pipe. With this assumption the effective length of the pulse-jet is L^*.

At the intake or reed-valve end, $x = 0$, suitable approximations for the boundary conditions are apparently not so obvious, and further experimental work is required to provide reliable relations (see Section 4). Since the usual reed valve openings permit flow through at most a small fraction of the cross-section area at $x = 0$ (see Fig. 3) the external effective ram pressure, p_1, probably remains fairly constant during a cycle. Presumably when the internal pressure p at $x = 0$ exceeds an empirically determinable value p_3 (depending on p_1 and on the constants and adjustment of the reed valves) the valves will stay shut and the end condition will be

(16) $$u = 0 \quad \text{when} \quad p > p_3 \quad \text{at} \quad x = 0+.$$

(closed valve condition)

When $p < p_3$ the valve openings will presumably act as nozzles with the varying flow through them satisfying a suitably generalized "pressure-head loss" relation(13, possibly of the form

(17) $$h(p,u,p_1) = 0 \quad \text{for} \quad p < p_3 \quad \text{at} \quad x = 0+.$$

If there were either no or steady combustion, then p and u in (17) would have steady values, say p_o and u_o. If the actual oscillations in the pulse-jet are regarded as perturbations of the steady state (as will be explained in (iv) below) then at $x = 0+$

(18) $$p = p_o + \Delta p \quad \text{and} \quad u = u_o + \Delta u$$

where Δp and Δu are the perturbations of the steady values p_o and u_o respectively. For sufficiently small Δp and Δu (18) in (17) yields the following approximate boundary condition:

(19) $$\Delta p = -z\Delta u \quad \text{for} \quad p < p_3 \quad \text{at} \quad x = 0+$$

(open valve condition) with

$$z = +[(\partial/\partial u_o)\, h(p_o,u_o,p_1)]/[(\partial/\partial p_o)\, h(p_o,u_o,p_1)]$$

a constant which might be fairly easily determined empirically if (19) is an appropriate boundary condition. (Note that (19) is equivalent to an impedance relation, which fact would be sufficient justification for it in the minds of many.) In the special case of initially steady flow it would be justified to take

(20) $$p = p_o \quad \text{and} \quad u = u_o \quad \text{at} \quad x = 0+$$

(i.e. just inside the head of the motor) until the time of arrival of compression waves from the initial region of ignition, and, assuming that the rise in p at that time is so rapid as to be considered instantaneous, thereafter to take (at $x = 0+$)

(20') $u = 0$ until the exhaust waves arrive at $x = 0$.

(iii) Combustion and flame speeds. Figure 5, based on an analysis(5 of high speed movies taken through holes in the side wall of a standard V.1 motor (under normal conditions of operation), indicates how, during a single cycle, the visible flames spread and persist after self-ignition at the position marked I in Fig. 5.

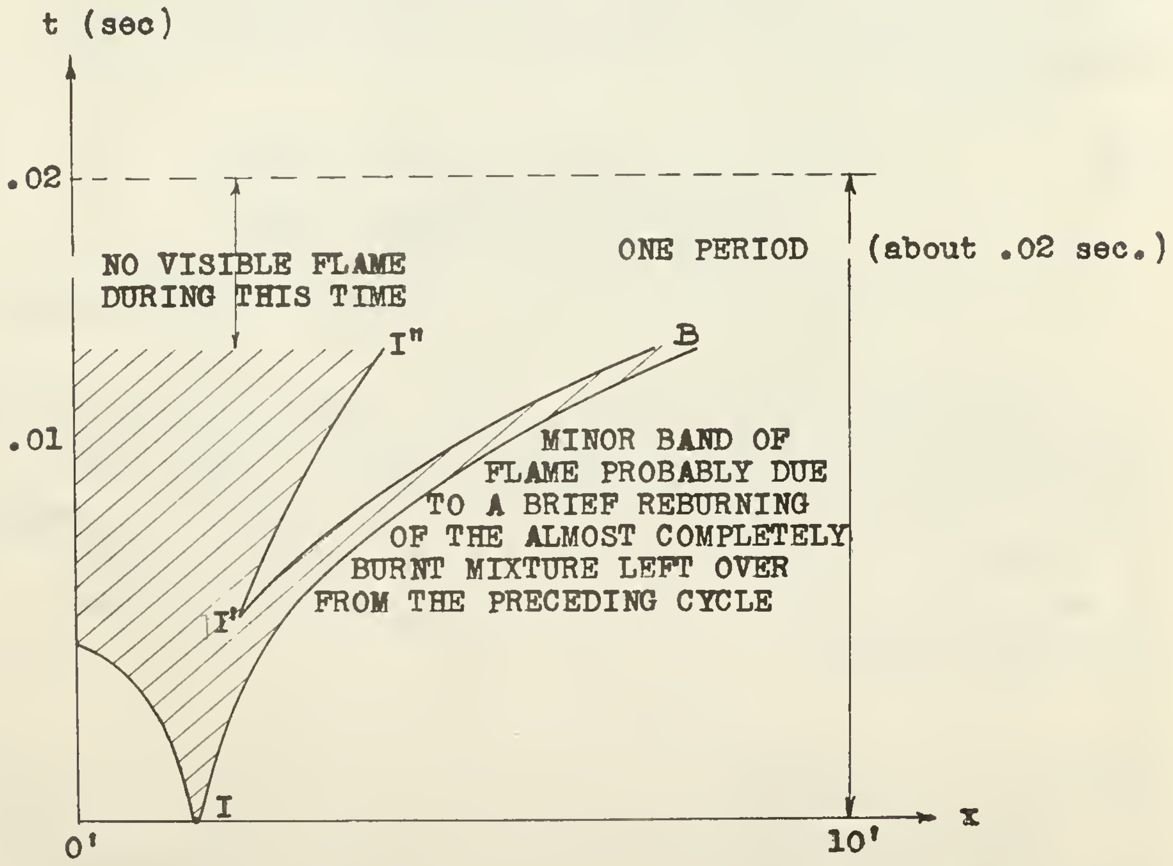

t (sec)
.02
.01
NO VISIBLE FLAME DURING THIS TIME
ONE PERIOD
(about .02 sec.)
I"
B
MINOR BAND OF FLAME PROBABLY DUE TO A BRIEF REBURNING OF THE ALMOST COMPLETELY BURNT MIXTURE LEFT OVER FROM THE PRECEDING CYCLE
I'
I
0'
10'
x

Fig. 5 Flame spread during a typical cycle of the V.1 motor. The shaded regions denote burning. (The gases remain hot but normally non-luminous after combustion.)

The point I of self-ignition of the gas probably marks the interface between the incoming fresh mixture and the almost completely burnt (but chain-carrier rich) mixture left over from the preceeding cycle. Due to the motion of the gases towards the exhaust end this interface moves with the particle speed $|u|$, as indicated by the curve II'I", while the minor band IB of reburning moves with a flame speed $|F|$ (relative to the motor) equal to the sum of the particle speed $|u|$ and the 'intrinsic flame speed' f relative to the gas. On the basis of this interpretation a numerical analysis of experimental curves corresponding to Fig. 5 indicates that both $|u|$ and f amount to a couple of hundred feet per second at I', and that the flame speed $|F|$ relative to the motor can be as large as 2000 ft/sec at B. (The local speed of sound at B is probably about 2500 ft/sec.) The maximum duration of combustion, as judged from luminosity, does not appear to exceed .014 sec. Doubtless the flame pattern would vary with the conditions of operation and with the geometry of the tube. In fact for the shorter 'Resojet' motor represented in Fig. 1(b) (page 1) the burning has been found to persist for a larger fraction of the period (.01 sec), and to be not so sharply outlined as that indicated in Fig. 5. The Resojet is therefore probably not very suitable for fundamental investigations which require consistently stable cycles.

To a first approximation the effective rate $\dot{E}$ (see p. 8) of release of heat energy in any given layer of the mixture is given by

$$\begin{aligned} \dot{E} &= 0 \quad \text{before and after combustion} \\ &= \Delta E/\Delta t \quad \text{during combustion,} \end{aligned} \tag{21}$$

where Δt is the duration of the burning and ΔE is the energy of combustion per gm of the mixture. (For a gasoline-air mixture in the weight ratio 1:16, with $\Delta t = .014$ sec, $\Delta E/\Delta t = 2.10^{12}$ cm^2/sec^3.) On the basis of chemical kinetics $\dot{E}$ might be expected to increase with both the temperature and the concentration of the combustible part of the mixture. During combustion these two agents would oppose one another and $\dot{E}$ might remain fairly constant. Experimental studies of the variations of $\dot{E}$ and the intrinsic flame speed $|F - u|$ are projected.

Various representations for p, in terms of time integrals involving $\dot{E}$, γ and t for any given particle (layer) of the mixture, are obtainable from (4):

If γ is constant

$$p\tau^{\gamma}\Big]_{t=0}^{t} = \int_{0}^{t} (\gamma-1)\,\dot{E}\,\tau^{\gamma-1}\,dt \qquad (22)$$

for a given particle. For the case of non-burning ($\dot{E} = 0$) at time t (22) yields the usual adiabatic relation: $p\tau^{\gamma} = \text{const.}$ If, as seems to be a good approximation, $C_p = R(C + G\theta)$, where R, C and G are constants, then

$$\frac{p}{p_1}\left(\frac{\theta}{\theta_1}\right)^{-C} = \exp\,(\theta - \theta_1)G - \int_{-\infty}^{t} \frac{\dot{E}}{R\theta}\,dt \qquad (22')$$

with $p\tau = R\theta$,

where p_1, θ_1 represent the values of p and θ outside the reed valves, and where $\dot{E} = 0$ until the particle begins to burn.

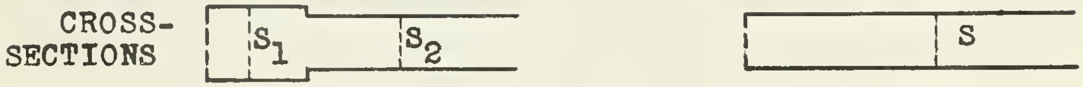
Relations like (22) and (22') are extensions of $p\tau^{-\gamma} = \text{const.}$ to cases involving combustion, and they may be useful if approximate values for the integrands are known. For example, the integrand in (22) might be regarded as constant during combustion of the particle, and zero otherwise.

(iv) Perturbation series for the equations. When the pulse-jet is first fired the combustion disturbs the initially steadily flowing (or resting) mixture, and for a certain interval the changes may be treated as fairly small perturbations (if violent detonations are avoided by use of suitable mixtures). Since the normal combustion in pulse-jets appears to be rather mild ($\dot{E}$ small) it might be expected that a couple of terms in perturbation series expansions(16 of the dependent variables in powers of a parameter such as the K defined below would provide good approximations

$$K = \text{maximum value of } [(\gamma - 1)\dot{E}]. \tag{23}$$

It is also convenient to introduce $I = (\gamma - 1)\dot{E}/K$ which therefore has the maximum value unity. The perturbation series in powers of K is given in Appendix II. If sufficient approximate data on the temperature and pressure distributions in a pulse-jet during its periodic oscillations were determined experimentally it would most probably be possible to deduce a mathematical representation which could be used to provide much better first terms in perturbation series of various convenient types. Analytical rather than purely numerical approximations are preferred for the investigation of the relative importance of the various physical parameters which govern the phenomena in pulse-jets.

The non-uniformity in the cross-section area S(x) of the pulse-jet leads to analytical complications which are minimized, in general, by treating the non-uniformity as a perturbation of the simple form (a) or (b) indicated in Fig. 6 (cf. Fig. 3).



(a) Limiting form of tube for the one-dimensionalized pulse-jet.

(b) Uniform tube for the pulse-jet.

Fig. 6 Approximate shape of pulse-jet tube.

In using the form indicated in Fig. 6(a) as the limit of a smooth one-dimensionalized form, continuity of $[S(x)u/\tau]$, $[u^2 + (2\gamma p\tau)/(\gamma - 1)]$ and $[p\tau^\gamma]$ across the discontinuity in $S(x)$, is implied by the hydro-thermodynamical equations (5)-(7). This is a mathematical rather than a strictly physical limit (which would involve three-dimensional considerations of reflections from constrictions). The mathematical limit probably provides a good approximation for the effects due to the fairly gradual constriction indicated in Fig. 3, but it is rather difficult to obtain analytical (rather than purely numerical) solutions of the hydro-thermodynamical equations by use of this limit. The form indicated in Fig. 6(b) is much simpler for use as a starting point for successive approximations to include the effects due to the mildly non-uniform cross-section of standard pulse-jets. An explicit formulation for perturbation series based on deviations from the form indicated in Fig. 6(b) is given in Appendix II.

Thus a double perturbation development of the equations is proposed: one perturbation being due to the effects of combustion, the other to the non-uniformity in the cross-section of the tube. In the simple example described in the following paragraphs only the first of these perturbations is used.

(v) Explicit treatment of an idealized case. An example is indicated in Fig. 7 below, which, in view of observations related to Fig. 5, probably is reasonably indicative for the main characteristics of the pulsating V.1 motor.

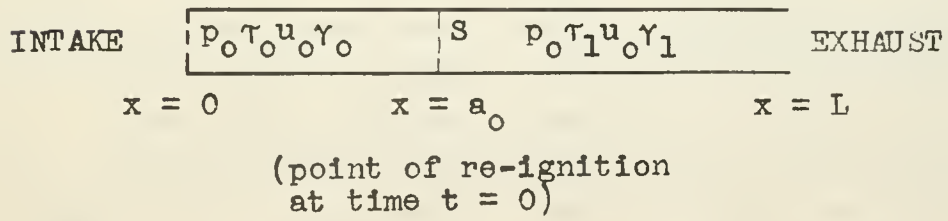

INTAKE
$p_0 \tau_0 u_0 \gamma_0$
S
$p_0 \tau_1 u_0 \gamma_1$
EXHAUST
$x = 0$
$x = a_0$
$x = L$
(point of re-ignition at time $t = 0$)

Fig. 7 Idealized example of a pulse-jet

Specifications:

Uniform tube with cross-section area S and length L.

Initial conditions:

At time $t = 0$: the part of the tube between $x = 0$ and $x = a_0$ (the point of re-ignition) contains a fresh, combustible, uniform mixture with pressure p_0, specific volume τ_0, temperature θ_0, flow velocity u_0, sound speed c_0 and ratio of specific heats γ_0; the rest of the tube (from $x = a_0$ to $x = L$) contains a hot, completely burnt gas with corresponding physical values p_0, τ_1, θ_1, u_0, c_1 and γ_1. These initial conditions are presumed to correspond roughly to the conditions in an actual V.1 motor at the instant of re-ignition.

Boundary conditions:

(cf. pages 14 and 15): At the exhaust end, $x = L$, $p = p_0$. At the intake end, $x = 0$: $p - p_0 = -(u - u_0)Z$ if $u > 0$ (open valve condition), or $u = 0$ if $p > p_0 + Zu_0$ (closed valve condition). Here Z is a constant which should be determined experimentally as indicated in Section 4, but which, on the basis of generalized forms of the simple Bernoulli equation (13 and (19) (page 15) is of the order of u_0/τ_0. We therefore assume $Z = u_0/\tau_0$.

Pseudo-Lagrange coordinates (m,t). (See Appendices I and III)

For convenience the mass flux leading to the above initial condition may be regarded as corresponding to a constant particle velocity u_0 throughout the tube until $t = 0$. Taking the reference time t_0 (see page 8) to be the corresponding time of arrival (at the intake) of the particle which reaches $x = L$ at $t = 0$, it follows that $t_0 = -L/u_0$ and that

$$(24)\quad m = \begin{cases} (L + u_0 t - x)S/\tau_1 & \text{if } x > a_0 + u_0 t \\ (L - a_0)S/\tau_1 + (a_0 + u_0 t - x)S/\tau_0 & \text{if } x < a_0 + u_0 t \end{cases}$$

until the disturbance due to the burning (which starts at $x = a_0$ at $t = 0$) reaches the position x concerned. The general relation connecting m, x and t is discussed in Appendix III.

Combustion and flame speed conditions.

The combustion term I defined below equation (23) (page 19) and concerned in (7), (8) and (11) (page 11), is here assumed to have the following values:

$$(25) \qquad I = \begin{cases} 1 & \text{during combustion of a particle} \\ 0 & \text{before and after combustion of a particle.} \end{cases}$$

The flame speed velocity is assumed to be such that if $m = m(t)$ is the pseudo-Lagrange coordinate of the flame front in the fresh mixture

$$(26) \qquad \frac{d}{dt} m(t) = \dot{m} = \text{const.}$$

This assumption amounts to regarding the flame spread as analagous to the successive firing of fire-crackers connected in line by fuses of uniform length. Experimental data is required to determine how good these assumptions may be. After $t = 0$ until the flame is extinguished (cf. Fig. 5, page 16) the burning extends, according to (26) and (24), from the particle with $m = (L - a_0)S/\tau_1$ ($= m_0$, say) to the particle with $m = m_0 + \dot{m}\, t$.

Explicit solutions of the pseudo-Lagrangian hydro-thermo-dynamical equations (11) - (13) subject to the above conditions are given to first order terms in Appendix III, for the first half of a cycle. The calculations are not carried further since physical data are lacking.

Numerical example. Using the formulas given in Appendix III, and the following typical numerical values for the physical quantities involved, the results shown in the graphs in Figures 8 and 9 were obtained:

$$K = 5.85 \cdot 10^{11} \mathrm{cm}^2/\mathrm{sec}^3, \ \gamma_o = 1.4, \ c_o = 3 \cdot 10^4 \mathrm{cm.sec}$$

$u_o = .2c_o$, $f = .3c_o$ (based on an analysis of Fig. 5, page 16)

$$Z = u_o/\tau_o, \ \tau_1/\tau_o = 7, \ \gamma_1 = 1.2$$

$c_1 = 2.5c_o$ (based on an absolute temperature $\theta_1 = 7\theta_o$)

$a_o = 45$ cm (1.5 ft), $L = 300$ cm (10ft) (cf. Fig. 1, page 1)

Although Fig. 8 corresponds to only a first approximation for propagation phenomena in the pulse-jet, it provides a good framework for the analysis of actual motions in the jet. Thus C_1 and C_2, for example, represent compression wave fronts which are reflected from the open exhaust end as rarefaction wave fronts R_1 and R_2. R_1 is reflected from the closed end as a rarefaction wave-front, following closely behind the rarefaction wave-front R_3 which is produced when the flame front reaches the closed intake valves. These rarefaction wave-fronts will in turn be reflected from the open end as compression waves. Judging from Fig. 5, these rarefaction waves appear to extinguish the flames as they pass through the region of burning (at a time corresponding roughly to the upper limit of Fig. 8); and the last mentioned compression waves probably stimulate the re-ignition automatically at, say, $t = 20$ ms. Fig. 9 indicates that the valves reopen at about $t = 10.2$ ms., allowing a fresh gas mixture to enter the pulse-jet.

The right side of the flame region in Fig. 8 indicates

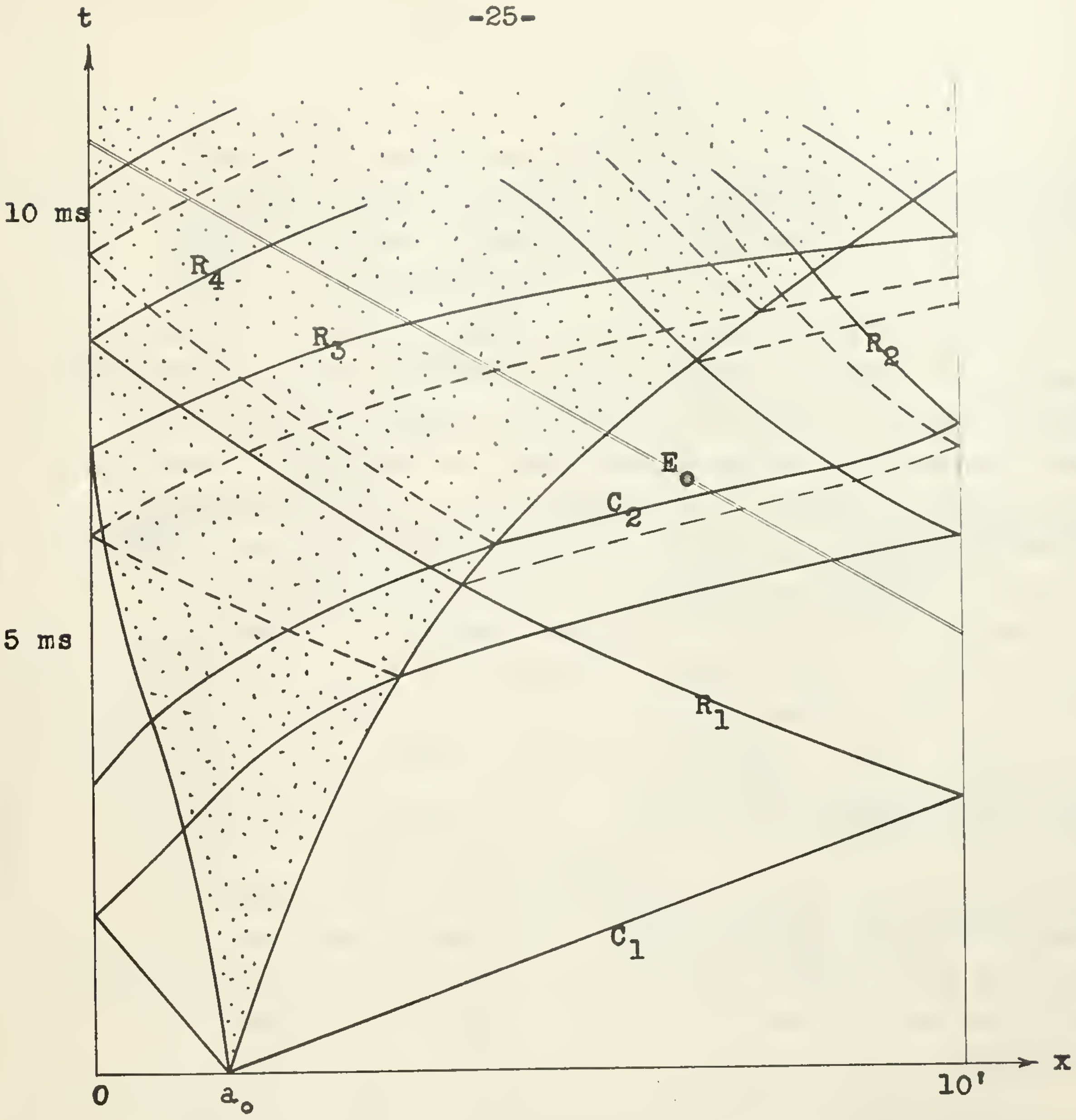


Fig. 8

Diagram for the first approximation to the wave fronts (characteristics) and flame region relative to the pulse-jet tube of Fig. 7. The first disturbances, due to the ignition at $x = 1.5$, reach the intake valves at time $t = 1.9$ ms, and the exhaust end at $t = 3.1$ ms. The intake valves close at time $t = 3.3$ ms, and reopen at time $t = 10.2$ ms. The flame front reaches the intake at $t = 7.2$ ms; the shading indicates the flame region. The important wave fronts are indicated by solid lines; the dotted lines represent relatively weak reflections from the interface between the previously burnt and the fresh gases.

the actual path of the interface particles. The rapid acceleration of these and the adjacent exhaust gases is notable (see also Fig. 9b) and probably makes $u_o + Kx_t^*$ (see Appendix III) a rather poor approximation for the gas velocity for values of x and t lying above the double line E_o in Fig. 8. (Higher approximations can be calculated without encountering too complicated forms, but have not been included since the values (and the nature) of some of the parameters used in this example are based on guesses rather than reliable data.) It is, for example, entirely unclear just how and when the flame is extinguished. The indicated motion of the flame region is in qualitative agreement with that actually observed in the early part (say the first third) of the cycle shown in Fig. 5 for the V.1 motor. For a comparison between theory and observation at later stages of the cycle it would appear advisable to modify the perturbation scheme at some point. A possible procedure would be to use Riemann's(14 exact integral representations of solutions of (5) and (6) subject to (3), starting with values given by the present first approximation on the above mentioned line E_o in Fig. 8. Graphical and numerical methods are also available(15. (See App. IV.)

In Fig. 9a an interesting feature is the considerable difference between the pressures at the intake end and at the interface between the fresh and previously burnt mixture. This difference suggests that the assumption (used in previous theories(2,(5) that the state of the gas is effectively uniform at all times in the combustion chamber is of doubtful validity.

It should also be noted that the pressure curve at the intake end for the first half of the cycle, as indicated in Fig. 9a, is in satisfactory agreement with rough observations.(1,(2

In Fig. 9a, the rapid drop in p/p_o at the intake valves after the flame front arrives is notable. Since the pressure

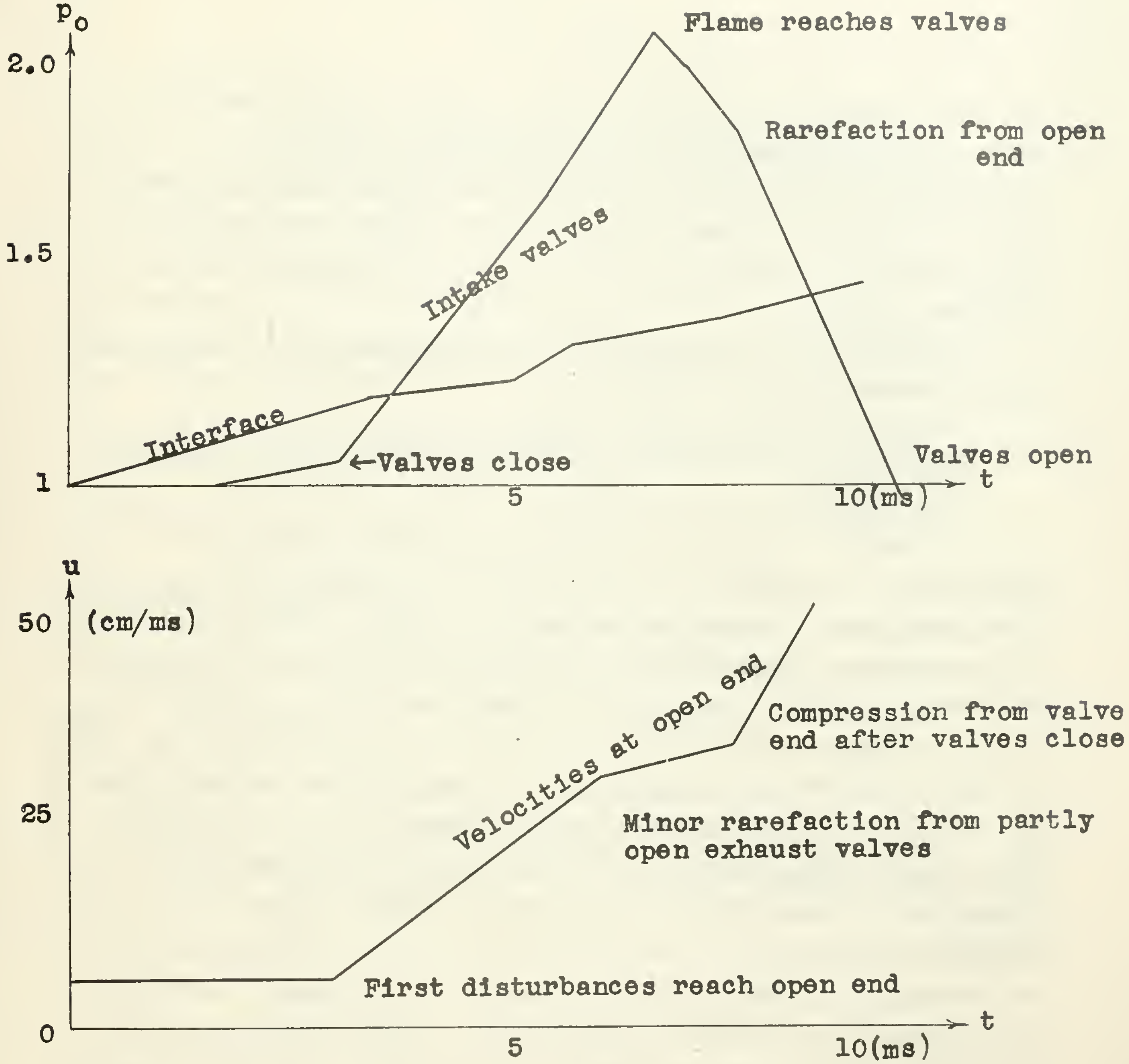


Fig. 9a,b.

Graphs for pressures and velocities (first approximation).

at the valves is essentially what drives the pulse-jet, this phenomenon suggests that the thrust might be considerably increased by keeping the flame away from the exhaust end for a longer time - for example by having a larger amount of combustible gas in the pulse-jet at the time of firing.

The rapid acceleration of the exhaust gases, mentioned above, is indicated in Fig. 9b.

4. Concerning the nature of desired experimental data on pulse-jets.

In order to make a reasonably accurate analysis of pulse-jets in terms of a mathematical framework such as that indicated in the preceding paragraphs, it is necessary to secure more fundamental physical data than is available at present. It is hoped that investigations, projected or in progress at N.R.L., N.A.C.A., Aerojet Corporation and N.Y.U., will be expanded so as to provide the desired data. In this connection the following comments may be helpful: High speed movies (2000 or more frames per second, both direct and shadowgraphic pictures): Judging from the preliminary movies taken through windows (or glass walls) in pulse-jets, much information can be obtained relatively easily by this method. Direct high speed movies can show (quantitatively) the motions of the flames from the time of re-ignition until the time of extinction: on the other hand, if a sodium compound is added to the fuel to make the hot gas glow, a measure of the turbulent motions of the hot gases may be obtained. Shadowgraphic high-speed movies may be made by directing sunlight through two transparent sides of a pulse-jet and then onto an otherwise shaded screen which can be photographed. Such shadowgraphic movies can show the turbulent motions of the gases as they move from the intake down the tube, and quantitative information may thus be obtained on the one-dimensionalized velocities involved. It would be desirable to obtain correlated direct and shadowgraphic pictures in order to distinguish between flame speeds and particle speeds, and it would furthermore be desirable to use a mirror system to permit the viewing (on the same films) of the motions of the intake valves. Ultimately it would be desirable to correlate the above observations with temperature and pressure measurements. (It might be remarked here that high speed movie studies of flame propagation in tubes under controlled conditions of ignition and turbulence, are under way at N.Y.U.)

Intake valve conditions. It would be of value to investigate experimentally the nature of the boundary condition at the reed valves and to see, in particular, if relation (19) page 15 is adequate. It should be relatively simple by means of Pitot gauges, to determine the relation among the ram pressure and the velocity and pressure just inside the intake end of the motor under steady flow conditions in a wind tunnel (without combustion). The question of how pulsing flow might modify such a relation because of dynamical factors would probably be fairly hard to answer experimentally. Possibly the dynamical effects might be evaluated by the use of shadowgram movies to measure effective (variable) speeds of the gas (delineated by mass movement of small vortices), coupled with oscillograms of the "static" pressures, a reciprocating pump being employed to produce an "artificially" pulsed (cool) exhaust.

Pressure and temperature oscillograms. Standard pick-ups for measuring variable ("static") pressures and variable temperatures in gases are apparently subject to large perturbations under the conditions of large and rapid mechanical and thermal oscillations encountered in pulse-jets. An important problem is therefore to design, construct and calibrate suitable measuring equipment. Some progress along these lines has been made, in particular by Aerojet and by the AMG-NYU in collaboration with Ensign Edelman of BuAer. Various methods for avoiding the effects of variation of the elastic "constants" with temperature in pressure diaphragms, have been incorporated in the designs. Photo-electric scanning is now being investigated in connection with spectroscopic methods for recording variable temperatures in hot gases. After such measuring devices have been precisely calibrated under conditions simulating those in pulse-jets it would be desirable to secure oscillograms of the pressures and temperatures simultaneously at several representative positions in pulse-jet motors. This information would permit, in particular, determination of the energy release rate $\dot{E}$ (cf. equations (4), (7), (8) and (11)) and of variables, other than velocities, involved in initial and end conditions. (More convenient than oscillograms would be movies of strings of glow lamps suspended above the motor and in view of the high speed cameras being used as described in a preceding paragraph. Circuits can be arranged to make the number of lamps glowing in a string at each instant proportional to the pressure or temperature signal concerned.)

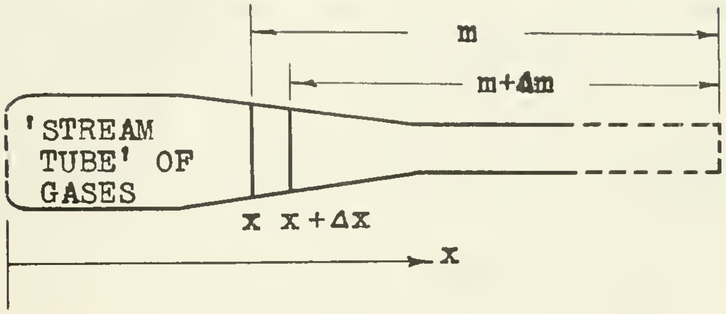
General data required for comparison of theory and actual performance:

(i) Dimensions of components of motor. (ii) Constants involved in intake (reed valve) boundary condition. (iii) Rate of flow of fuel (preferably oscillogram record). (iv) Combustion and flame constants for the fuel under the turbulence conditions involved (determined by pressure, temperature, particle velocity (high speed movies) and flame pattern observations in pulse-jets in general).

APPENDIX I

Derivation of the equations in (m,t).

(i) At time t, the mass contained between the layers at x and at $x + \Delta x$ is $-\Delta m$. Hence



$$-\Delta m = S(x)\cdot \Delta x/\tau \ .$$

Thus, in the limit, we have the continuity equation

$$x_m = -\tau/S(x). \tag{12}$$

(ii) The equation of motion, Force = mass x acceleration, for a small layer of gas is

$$S(x)[p(m) - p(m+\Delta m)] = -(\Delta m)x_{tt}.$$

Hence, in the limit,

$$S(x)p_m = x_{tt}. \tag{13}$$

(iii) Since m is constant for a given particle (layer), the internal energy equation

$$(\gamma - 1)\dot{E} = \gamma p \tau_t + \tau p_t \tag{11}$$

remains formally the same as in Lagrangian coordinates.

APPENDIX II

Perturbation Series

1. Perturbation series in $\varepsilon = \max\left|\frac{S(x)-S_o}{S(x)}\right|$.

The constant S_o is some conveniently chosen mean value of $S(x)$. We suppose that the cross-section of the pulse-jet tube does not vary too much, and hence that ε is a small number. For the German VI motor, $\varepsilon \sim .3$. A power series development of x, p, and τ in terms of ε is then assumed:

$$
\begin{aligned}
x &= X + \varepsilon X^* + \varepsilon^2 X^{**} + \ldots \\
p &= P + \varepsilon P^* + \varepsilon^2 P^{**} + \ldots \\
\tau &= T + \varepsilon T^* + \varepsilon^2 T^{**} + \ldots
\end{aligned} \tag{26}
$$

where all quantities except the small constant ε are functions of m and t. Furthermore we may define $S^*(x)$ so that

$$S(x) = S_o + \varepsilon S^*(x).$$

Then

$$
\begin{aligned}
S(x) &= S_o + \varepsilon S^*(X + \varepsilon X^* + \ldots) \\
&= S_o + \varepsilon S^*(X) + \varepsilon^2 X^* S^*_X(X) + \ldots
\end{aligned} \tag{27}
$$

We now substitute (26) and (27) into (11), (12) and (13) and equate coefficients of like powers of ε. This furnishes the following sets of equations.

$$E_o \left\{ \begin{aligned} T &= - S_o X_m \\ X_{tt} &= S_o P_m \\ (\gamma - 1)\dot{E} &= \gamma PT_t + TP_t \end{aligned} \right. \qquad \text{(terms not involving } \varepsilon\text{)}$$

$$E_1 \left\{ \begin{aligned} T^* &= -[S^*(X)X_m + S_o X^*_m] \\ X^*_{tt} &= S^*(X)P_m + S_o P_m \\ \gamma(PT^*_t + P^*T_t) &+ (TP^*_t + T^*P_t) = 0 \end{aligned} \right. \qquad \text{(coefficients of } \varepsilon\text{)}$$

$$E_2 \left\{ \begin{aligned} T^{**} &= -[S_o X^{**} + S^*(X)X^*_m + X_m X^* S^*_X(X)] \\ X^{**}_{tt} &= S_o P^{**}_m + S^*(X)P^*_m + P_m X^* S^*_X(X) \\ \gamma(PT^{**}_t &+ P^*T^*_t + P^{**}T_t) + (TP^{**}_t + T^*P^*_t + T^{**}P_t) = 0 \text{ , etc.} \end{aligned} \right. \qquad \text{(coefficients of } \varepsilon^2\text{)}$$

We note that E_o is a non-linear system; but once E_o is solved so that X, T, and P are known functions of m and t, the successive systems E_1, E_2, etc. are each linear and may be solved in turn for X^*, P^*, T^*; X^{**}, P^{**}, T^{**} etc. The system E_o corresponds to a pulse-jet of uniform cross-section S_o. Since it is non-linear we seek a second perturbation series for its solution.

2. Perturbation series for E_o in $K = \max(\gamma - 1)\dot{E}$.

We suppose that a power series development of X, P etc. in powers of K is possible so that

$$\text{(28)} \qquad \begin{aligned} X &= \chi + K\chi^* + K^2\chi^{**} + \ldots \\ P &= \wp + K\wp^* + \ldots \\ T &= \mathcal{T} + K\mathcal{T}^* + \ldots \\ \gamma &= \mathfrak{y} + K\mathfrak{y}^* + \ldots \end{aligned}$$

We define I by the equation

$$I = (\gamma - 1)\dot{E}/K ,$$

so that

$$(29)\qquad \begin{aligned} I &= \delta + K\delta^*(P,T,t) \\ &= \delta + K\delta^*(\mathcal{P},\mathcal{T},t) + K^2(\mathcal{P}^*\delta^*_{\mathcal{P}} + \mathcal{T}^*\delta^*_{\mathcal{T}}) + \ldots \end{aligned}$$

and $\delta = 1$ or 0 according as the particle is burning or not. Proceeding as before, we substitute (28) and (29) into E_o, p. 34, and equate coefficients of like powers of K. We obtain

$$E'_o \quad \begin{cases} \mathcal{T} = -\, s_o \chi_m \\ \chi_{tt} = s_o \mathcal{P}_m \\ \mathcal{G}\mathcal{P}\mathcal{T}_t + \mathcal{T}\mathcal{P}_t = 0 \end{cases} \qquad \text{(terms not involving K)}$$

$$E'_1 \quad \begin{cases} \mathcal{T}^* = -\, s_o \chi^*_m \\ \chi^*_{tt} = s_o \mathcal{P}^*_m \\ \delta = \mathcal{G}(\mathcal{P}\mathcal{T}^*_t + \mathcal{P}^*\mathcal{T}_t) + \\ \qquad \mathcal{G}^*(\mathcal{P}\mathcal{T}_t) + \mathcal{T}\mathcal{P}^*_t + \mathcal{T}^*\mathcal{P}_t \end{cases} \qquad \text{(coefficients of K)}$$

etc.

As before, only E'_o is non-linear, the succeeding systems being all linear. The system E'_o corresponds to a pulse-jet of uniform cross-section with no burning -- i.e. is merely a set of relations which the initial state of the system, given by $\mathcal{P}$, $\mathcal{T}$ and χ, must satisfy at the time $t = 0$ of firing. We are hence not required to solve E'_o, but merely to check that the initial conditions (i.e. the functions $\mathcal{P}$, $\mathcal{J}$ and $\mathcal{X}$) which we postulate, are in agreement with these relations.

APPENDIX III

Explicit Solution of an Idealized Case

1. General Solution.

We consider the idealized pulse-jet, described on pages 21 and 22, with uniform cross-section S and length L.

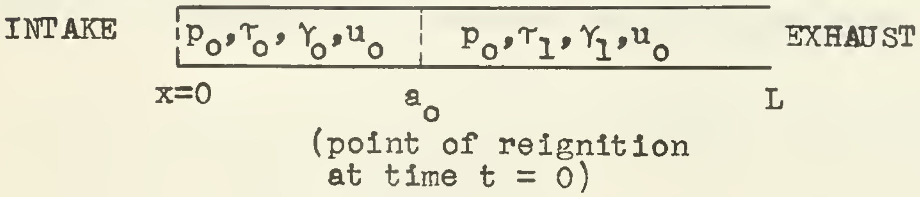

INTAKE
$p_0, \tau_0, \gamma_0, u_0$
$p_0, \tau_1, \gamma_1, u_0$
EXHAUST
x=0
a_0
L
(point of reignition at time t = 0)

Fig. 10 Idealized pulse-jet.

At the time $t = 0$ of firing, it is assumed that the gas in the tube is in motion with constant velocity u_0 and has constant pressure p_0; and that a fresh combustible mixture with constant specific volume τ_0 and adiabatic exponent γ_0 extends from $x = 0$ to $x = a_0$, the point of firing, while completely burnt gas with constant specific volume τ_1 and adiabatic exponent γ_1 extends from $x = a_0$ to $x = L$. Since the cross-section is uniform, $S(x) = S_0 = S$, and $\varepsilon = 0$. Hence, referring to (26), $x = X$, $p = P$, and $\tau = T$. Accordingly, only the perturbation series in powers of K is to be considered. (In considering a pulse-jet of non-uniform cross-section, the present solution would of course be the starting point for the perturbation series in ε.)

From the prescribed initial conditions we have

$$\mathcal{T} = \begin{cases} \tau_o & 0 \leq x \leq a_o \\ \tau_1 & a_o \leq x \leq L \end{cases}$$

$$\mathcal{P} = p_o$$

$$\mathcal{G} = \begin{cases} \gamma_o & 0 \leq x \leq a_o \\ \gamma_1 & a_o \leq x \leq L\ . \end{cases}$$

We also have, as is clear from inspection of Fig. 11,

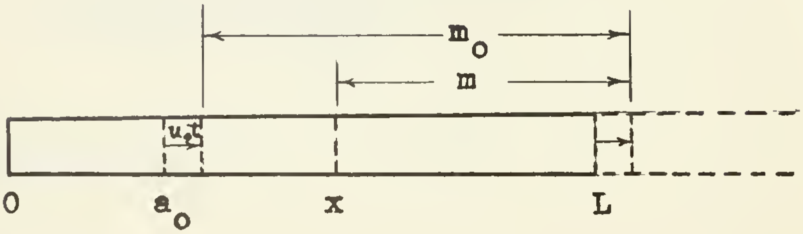


Fig. 11 Relation between m and x at time t.

$$m = \begin{cases} (L + u_o t - x)S/\tau_1 & a_o \leq x \leq L \\ m_o + (a_o + u_o t - x)S/\tau_o & 0 \leq x \leq a_o\ , \end{cases}$$

where m_o is the mass of burnt gas,

$$m_o = (L - a_o)S/\tau_1\ .$$

Hence

$$\mathcal{X} = \begin{cases} L + u_o t - m\tau_1/S & 0 \leq m \leq m_o \\ a_o + u_o t - (m - m_o)\ \tau_o/S & m > m_o \end{cases}$$

We observe that $\mathcal{T}$, $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{X}$ so prescribed satisfy the system E_0'.

Since we are taking m and t as independent variables, we must refer the boundary condition at $x = 0$ and $x = L$ to the corresponding points in the (m,t) plane. Now $x = \mathcal{X} + K\mathcal{X}^* + \ldots$, and it follows that the value of m for the particle which reaches $x = L$ at time t will be given by

$$m = u_0 St/\tau_1 + [\text{terms involving } K].$$

For the first approximation, then, we replace the actual boundary $x = L$ by the approximation

$$m = u_0 St/\tau_1$$

Similarly, for the boundary $x = 0$ (the intake valve end) we take

$$m = \begin{cases} m_0' + u_0 St/\tau_1 & \text{until the valves close} \\ m_2' & \text{after the valves close} \end{cases}$$

where m_0' is the total mass of gas in the tube at time $t = 0$,

$$m_0' = [(L - a_0)/\tau_1 + a_0/\tau_0]S,$$

and

$$m_2' = m_0' + u_0 St_2'/\tau_0,$$

t_2' being the time of closure of the valves.

We assume that the flame-front is propagated from particle to particle at a constant rate f -- independent of the motion or density of the gas. If $m_f(t)$ is the value of m for the particle which begins burning at time t, we have

$$m_f(t) = m_o + fSt/\tau_o = m_o + \dot{m}t \ ,$$

with $\dot{m} = fS/\tau_o$.

In the (m,t) plane, then, the region in which we wish to solve the system E_1' is as indicated in Fig. 12 (below).

The system E_1' can now be written in specific form, including the initial conditions:

$$E_1' \begin{cases} \mathcal{X}^*_{tt} = S\,\mathcal{P}^*_m & \\ \mathcal{J}^* = -\,S\,\mathcal{X}^*_m & \\ \gamma_o p_o\,\mathcal{J}^*_t + \tau_o\,\mathcal{P}^*_t = \mathcal{D} & m_o < m \\ \gamma_1 p_o\,\mathcal{J}^*_t + \tau_1\,\mathcal{P}^*_t = 0 & m < m_o \end{cases}$$

where, at any time t,

$$\mathcal{D} = \begin{cases} 1 & m_o < m < m_f \\ 0 & m > m_f \ . \end{cases}$$

The first equation in E_1' can be satisfied identically by introducing a function V(m,t), analogous to a stream-function, such that

$$\mathcal{X}^*_t = \text{const} \cdot S\,V_m$$

$$\mathcal{P}^* = \text{const} \cdot V_t$$

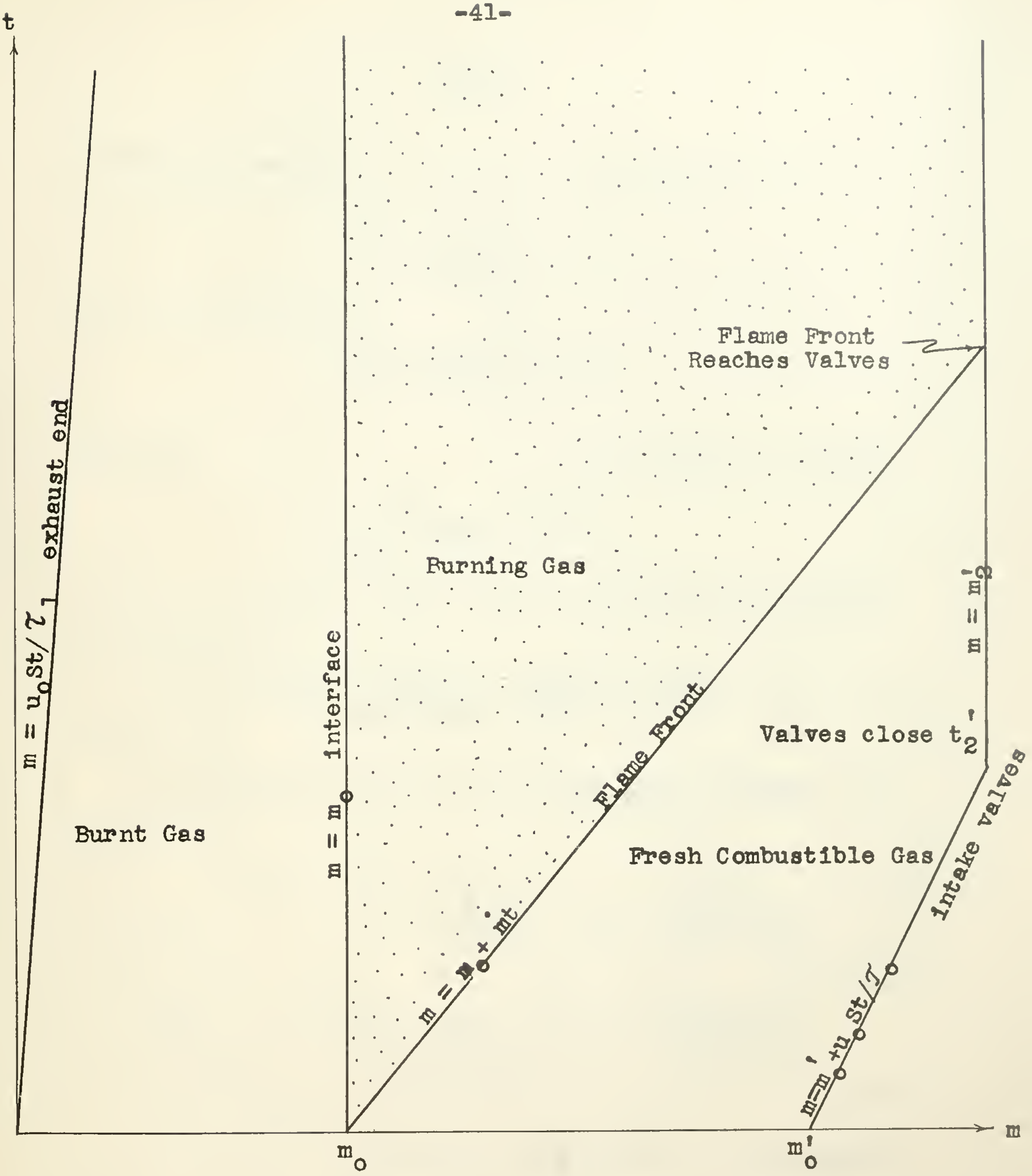


Fig. 12 Region in (m,t) plane corresponding to pulse-jet.

It is convenient to choose the constant so that

$$\chi_t^* = \frac{s}{2\tau_o(a^2-\dot{m}^2)} V_m$$

$$\sigma^* = \frac{1}{2\tau_o(a^2-\dot{m}^2)} V_t ,$$

where a is given by

$$a^2 = \gamma_o p_o s^2 / \tau_o .$$

In terms of V, we have

$$\mathcal{T}_t^* = - s\chi_{mt}^* = \frac{-s^2}{2\tau_o(a^2-\dot{m}^2)} V_m ;$$

the system E_1' then reduces to the pair of equations

$$\frac{-\gamma_o p_o s^2}{2\tau_o(a^2-\dot{m}^2)} V_{mm} + \frac{\tau_o}{2\tau_o(a^2-\dot{m}^2)} V_{tt} = \delta \qquad m > m_o$$

$$\frac{-\gamma_1 p_o s^2}{2\tau_o(a^2-\dot{m}^2)} V_{mm} + \frac{1}{2\tau_o(a^2-\dot{m}^2)} V_{tt} = 0 \qquad m < m_o .$$

These may be conveniently written as

(31) $$V_{tt} - a^2 V_{mm} = 2(a^2-\dot{m}^2)\delta \qquad m > m_o$$

(32) $$V_{tt} - b^2 V_{mm} = 0 \qquad m < m_o ,$$

where

$$b^2 = \gamma_1 p_o S^2/\tau_1 .$$

Since the sound speed c is given by

$$c^2 = \gamma p/\rho = \gamma p \tau ,$$

we have

$$a = c_o S/\tau_o , \; b = c_1 S/\tau_1 ,$$

c_o and c_1 being the sound speeds in the fresh and burnt mixtures respectively.

Equations (31) and (32) are hyperbolic partial differential equations whose general solution may be written in the form

$$(33) \quad V = \begin{cases} f(m + at) + g(m - at) - (m - m_o - \dot{m}t)^2 \mathscr{L} & m > m_o \\ F(m + bt) + G(m - bt) & m < m_o \end{cases}$$

The function $-(m - m_o - \dot{m}t)^2 \cdot \mathscr{L}$ ($\mathscr{L}$ defined on page 40) is a particular integral of (31), which vanishes together with its first partial derivatives along the flame front $m - m_o - \dot{m}t = 0$.

The first approximation for the pressure and velocity may now be written in terms of the function V,

$$(34) \qquad p = p_o + \frac{K}{2\tau_o(a^2 - \dot{m}^2)} V_t$$

$$(35) \qquad u = u_o + \frac{KS}{2\tau_o(a^2 - \dot{m}^2)} V_m .$$

It remains only to determine the function V (i.e. f, g, F and G) in such a way as to satisfy the boundary conditions.

Boundary Conditions.

(i) Open end. At the open end, we should have $p = p_o$, i.e.

$$V_t = 0 \quad \text{at } m = u_o St/\tau_1 . \tag{36}$$

(ii) Intake end. At the intake valves, we should have $p - p_o = -z(u - u_o) = -u_o(u - u_o)/\tau_o$, until the valves close (at $t = t_2'$, when $u = 0$); and thereafter $u = 0$. Hence

$$V_t = -Su_oV_m/\tau_o \qquad \text{at } m = m_o' + u_oSt/\tau_o , \; t < t_2' \tag{37}$$

$$V_m = -2\tau_o u_o(a^2 - \dot{m}^2)/KS \qquad \text{at } m = m_2' , \; t > t_2' \tag{38}$$

We further require that V, V_m and V_t be continuous throughout -- and in particular at the interface, $m = m_o$. At $t = 0$; V, V_m and V_t are zero.

The characteristics for equations (31) and (32) are straight lines of the form

$$m \pm at = \text{const.} \qquad m > m_o$$

$$m \pm bt = \text{const.} \qquad m < m_o ,$$

In Fig. 13 below, we have redrawn Fig. 12, but included all relevant characteristics.

Referring to Fig. 13 and (33), it may be shown that V will have the form

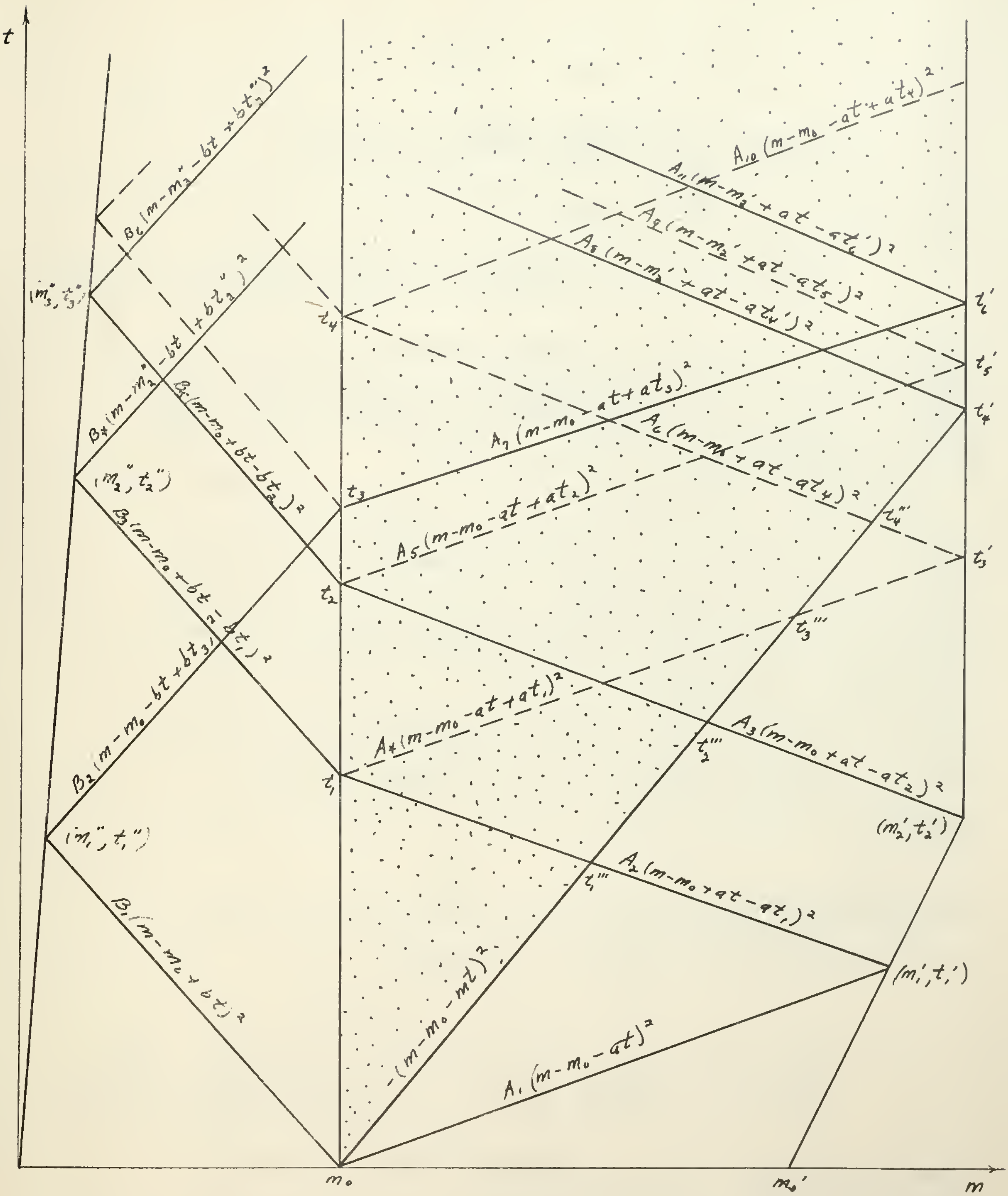


Fig. 13 Region in (m,t) plane corresponding to pulse-jet including relevant characteristics. Cf. Fig. 12.

$$(39)\quad V = \begin{cases} -\mathscr{d}\cdot(m - m_o - \dot{m}t)^2 + A_1^*(m - m_o - at)^2 + \\ \qquad A_2^*(m - m_o + at - at_1)^2 + \ldots & m > m_o \\ B_1^*(m - m_o + bt)^2 + B_2^*(m - m_o - bt + bt_3)^2 + \ldots & m < m. \end{cases}$$

where the A^* 's and B^* 's are zero for values of m and t (in Fig. 13) below the corresponding characteristic line, and have constant values A_1, A_2, B_1, B_2 etc. above the corresponding line. This form is suggested by the following facts:

(i) that V, V_m and V_t must be continuous across all characteristics, (ii) tho particular solution -- $-(m - m_o - \dot{m}t)^2\cdot\mathscr{d}$ is quadratic in m and t, and all terms must match along the boundaries. The solution is of course unique, so that all that need be done to justify (39) is to verify that the constant A's and B's can be determined in such a way as to satisfy the boundary conditions. The quadratic form may also be deduced directly by use of (33), but this is rather tedious; we will assume the form indicated in (39) and proceed to the determination of the constants.

Determination of A's and B's.

(i) Interface. Along $m = m_o$, V and its derivatives are required to be continuous. We have

$$V = \begin{cases} A_1\cdot(m - m_o - at)^2 - (m - m_o - \dot{m}t)^2\cdot\mathscr{d} & m > m_o \\ B_1\cdot(m - m_o + bt)^2 & m < m_o \end{cases}$$

Hence setting $m = m_o$ and remembering that $\mathscr{d} = 1$ along $m = m_o$, we have

$$A_1 a^2 t^2 - \dot{m}^2 t^2 = B_1 b^2 t^2$$

$$A_1 at - \dot{m}t = B_1 bt \;.$$

We may solve for A_1 and B_1, obtaining

$$A_1 = \dot{m}(b + \dot{m})/a(a + b)$$

$$B_1 = \dot{m}(a - \dot{m})/b(a + b)$$

Similarly,

$$A_4 = \frac{b-a}{b+a} A_2 = M_1 A_2$$

$$B_3 = \frac{2a^2}{b(a+b)} A_2 = M_2 A_2$$

$$A_5 = M_1 A_3$$

$$B_5 = M_2 A_3$$

$$A_7 = \frac{2b^2}{a(a+b)} B_2$$

$$A_{10} = \frac{b-a}{b+a} A_6 \;.$$

(ii) Exhaust end. At the exhaust end, just after the first wave arrives at $t = t_1''$, we have

$$V = B_1 \cdot (m - m_o + bt)^2 + B_2 \cdot (m - m_o - bt + bt_3)^2$$

$$= B_1 \cdot [(m - m_1'' + b(t - t_1''))]^2 + B_2 \cdot [m - m_1'' - b(t - t_1'')]^2 \;.$$

Substituting in (36), we obtain

$$V_t = 2b[B_1\cdot(b + u_o S/\tau_1) + B_2\cdot(b - u_o S/\tau_1)](t - t_1'') = 0 \ .$$

Hence

$$B_2 = \frac{b+u_o S/\tau_1}{b-u_o S/\tau_1} B_1 = - M_3 B_1 \ .$$

Similarly,

$$B_4 = - M_3 B_3 \ ; \ B_6 = - M_3 B_4 \ .$$

(iii) Intake valves. In the neighborhood of the intake valves, just after the first disturbance arrives at the valves but before they close, V is given by

$$V = A_1\cdot(m - m_o - at)^2 + A_2\cdot(m + at - m_o - at_1)^2$$

$$= A_1\cdot[(m - m_o' - a(t - t_1'))]^2 + A_2\cdot[m - m_o + a(t - t_1')]^2 \ .$$

Substituting in (37), we obtain

$$2a[- A_1\cdot(u_o S/\tau_o - a) + A_2\cdot(u_o S/\tau_o + a)](t - t_1')$$

$$= \frac{-2Su_o}{\tau_o} [A_1\cdot(u_o S/\tau_o - a) + A_2\cdot(u_o S/\tau_o + a)](t - t_1').$$

Hence,

$$A_2 = -[\frac{a-u_o S/\tau_o}{a+u_o S/\tau_o}]^2 A_1 \ .$$

After the valves close, at (m_2', t_2') we have

$$V = A_1\cdot(m - m_o - at)^2 + A_2\cdot(m - m_o + a(t - t_1))^2$$
$$+ A_3\cdot[(m - m_2' + a(t - t_2')]^2 .$$

Substituting in (38), we have

$$V_m = \text{const} + 2a[- A_1 + A_2 + A_3](t - t_2') = \text{const}.$$

Hence

$$A_3 = A_1 - A_2 .$$

Similarly, we obtain

$$A_4 = A_6 \text{ , } A_9 = A_5; \quad A_{11} = A_7; \quad A_8 = - \dot{m}/a .$$

This completes the determination of the A's and B's.

Determination of the t's.

The time t_2' corresponding to the closing of the valves is determined by the equation $u = 0$ at $m = m_o' + u_o St/\tau_o$. We have from (38),

$$V_m = - 2 \tau_o u_o (a^2 - \dot{m}^2)/KS ,$$

Hence

$$t_2' = t_1' + \frac{\tau_o u_o (a^2 - \dot{m}^2)(a+b)(a+u_o S/\tau_o)}{2KS\dot{m}(b+\dot{m})(a-u_o S/\tau_o)}$$

This may be written in terms of the quantities f, c_o, and c_1 as

time of closure of valves

$$t_2' = t_1' + \frac{u_o(c_o^2-f^2)(c_o\tau_1+c_1\tau_o)(c_o+u_o)}{2Kf(c_1\tau_o+f\tau_1)(c_o-u_o)} .$$

All other points (m,t) indicated in Fig. 13 are determined by elementary analytic geometry, as the intersections of given lines. We have:

(i) Intake valves:

first disturbances arrive

$$t_1' = \frac{a_o}{c_o-u_o} ,$$

flame-front arrives

$$t_4' = \frac{a_o+u_ot_2'}{f} ,$$

rarefaction from tail-pipe

$$t_6' = t_3 + \frac{a_o+u_ot_2'}{c_o} ,$$

miscellaneous reflections

$$t_3' = t_1 + \frac{a_o+u_ot_2'}{c_o} ,$$

$$t_5' = t_2 + \frac{a_o+u_ot_2'}{c_o} .$$

(ii) Flame front:

miscellaneous reflections cross flame-front

$$t_1''' = \frac{at_1}{\dot{m}+a} ; \quad t_2''' = \frac{at_2}{\dot{m}+a} ; \quad t_3''' = \frac{at_1}{a-\dot{m}} ; \quad t_4''' = \frac{at_4}{a+m}$$

(iii) Interface:

miscellaneous reflections cross interface

$$t_1 = 2t_1' \; ; \; t_2 = t_2' + \frac{a_0+u_0t_2'}{c_0}$$

$$t_3 = 2t_1'' \; ; \; t_4 = t_1 + 2\,\frac{a_0+u_0t_2'}{c_0} \, .$$

(iv) Exhaust end:

first disturbance arrives

$$t_1'' = \frac{L-a_0}{c_1+u_0}$$

subsequent reflections

$$t_2'' = t_1'' + \frac{c_1t_1}{c_1+u_0} \; ; \; t_3'' = t_3'' + \frac{c_1t_2}{c_1+u_0} \, .$$

We have thus completely determined the function V for the portion of the cycle corresponding to Fig. 13. From this function V, may be deduced the pressure, position, etc. of any particle at any time. In particular, we are interested in the following functions of the time

(i) the velocity of gases at the exhaust end

(ii) the pressure at the interface

(iii) the pressure at the intake valves

(iv) the position of the interface

(v) the position of the flame-front.

The first three of these quantities follow immediately from (33) and (34), the relations between the A's and B's and the formulation of V. We have (with each term being added to the terms above it after the time indicated in the inequality)

$$\text{(i)}\quad u = u_o + \frac{2KS(b+u_oS/\tau_1)}{\tau_o(a^2-\dot{m}^2)}\left\{\begin{array}{ll} \qquad 0 & t > 0 \\ + B_1(t - t_1'')\ , & t > t_1'' \\ + B_3(t - t_2'')\ , & t > t_2'' \\ + B_5(t - t_3'')\ , & t > t_3'' \\ \text{etc.} & \end{array}\right.$$

$$\text{(ii)}\quad \frac{p}{p_o} = 1 + \frac{\gamma_oKa^2}{c_o^2(a^2-\dot{m}^2)}\left\{\begin{array}{ll} + (A_1 - \dot{m}^2/a^2)t & t > 0 \\ + (A_2 + A_4)(t - t_1) & t > t_1 \\ + (A_3 + A_5)(t - t_2) & t > t_2 \\ + \ A_7(t - t_3) & t > t_3 \\ + (A_6 + A_{10})(t - t_4) & t > t_4 \\ \text{etc.} & \end{array}\right.$$

$$\text{(iii)}\quad \frac{p}{p_0} = 1 + \frac{\gamma_0 K a^2}{c_0^2(a^2-\dot{m}^2)} \begin{cases} 0 & t > 0 \\ +\left[A_1\left(1-\frac{u_0 S}{\tau_0 a}\right)+A_2\left(1+\frac{u_0 S}{\tau_0 a}\right)\right](t-t_1') & t > t_1' \\ +\ A_3(t - t_2') & t > t_2' \\ +\ 2A_4(t - t_3') & t > t_3' \\ +\ (A_8 - \dot{m}^2/a^2)(t - t_4') & t > t_4' \\ +\ 2A_5(t - t_5') & t > t_5' \\ +\ 2A_7(t - t_6') & t > t_6' \\ \text{etc.} & \end{cases}$$

The position $x(m,t)$ at time t of a particle, which corresponds to the mass-coordinate m, is given by

$$x(m,t) = x(m,0) + \int_0^t u(m,t)dt \ .$$

Using (35), this reduces to

$$\text{(39)}\qquad x(m,t) = x(m,0) + u_0 t + \frac{KS}{2\tau_0(a^2-\dot{m}^2)} \int_0^t V_m dt \ .$$

In particular, for the interface, $m = m_0$ and $x(m,0) = a_0$, so that (39) becomes

$$(iv)\ x(m,t) = a_o+u_ot + \frac{KSa}{2\tau_o(a^2-\dot{m}^2)}\begin{cases} (\dot{m}/a-A_1)t^2 & t > 0 \\ +\ (A_2-A_4)(t-t_1)^2 & t > t_1 \\ +\ (A_3-A_5)(t-t_2)^2 & t > t_2 \\ -\ A_7(t-t_3)^2 & t > t_3 \\ +\ (A_6-A_{10})(t-t_4)^2 & t > t_4 \\ \text{etc.} & \end{cases}$$

The mass-coordinate for the particle which is at the flame-front at time t is given by

$$m = m_o + \dot{m}t\ .$$

Substituting this value for m in (39) we obtain, for the position of the flame front, the expression

$$(v)\ x(m,t) = a_o+(u_o-f)t + \frac{KS}{2\tau_o a(a^2-\dot{m}^2)}\begin{cases} -\ A_1(\dot{m}-a)^2t^2 & t > 0 \\ +\ A_2(\dot{m}+a)^2(t-t_1''')^2 & t > t_1''' \\ +\ A_3(m+a)^2(t-t_2''')^2 & t > t_2''' \\ -\ A_4(\dot{m}-a)^2(t-t_3''')^2 & t > t_3''' \\ +\ A_6(\dot{m}+a)^2(t-t_4''')^2 & t > t_4''' \end{cases}$$

2. Numerical Example.

Using the values for the parameters listed on page 24, we obtain the following:

(i) Velocity of gases at tail pipe; in cm/ms :

$$u = 6 + 17\left\{\begin{array}{lll} 0 & t > 0 & \\ + .43(t - 3.1) & t > 3.1 & \text{(ms.)} \\ - .27(t - 6.6) & t > 6.6 & \\ + .86(t - 8.1) & t > 8.1 & \end{array}\right.$$

(ii) Pressure ratio p/p_o at interface:

$$\frac{p}{p_o} = 1 + \left\{\begin{array}{lll} .056t & t > 0 & \\ - .035(t - 3.7) & t > 3.7 & \text{(ms.)} \\ + .110(t - 5.5) & t > 5.5 & \\ - .096(t - 6.2) & t > 6.2 & \\ + .016(t - 8.1) & t > 8.1 & \end{array}\right.$$

(iii) Pressure ratio p/p_o at intake valves:

$$\frac{p}{p_o} = 1 + \left\{\begin{array}{lll} 0 & t > 0 & \\ + .039(t - 1.9) & t > 1.9 & \text{(ms.)} \\ + .211(t - 3.3) & t > 3.3 & \\ + .060(t - 5.9) & t > 5.9 & \\ - .390(t - 7.2) & t > 7.2 & \\ - .200(t - 7.6) & t > 7.6 & \\ - .192(t - 8.4) & t > 8.4 & \end{array}\right.$$

(iv) Position of the interface (cm)

$$x = 45 + 6t + .357 \begin{cases} .154t^2 & t > 0 \\ - .095(t - 3.7)^2 & t > 3.7 \quad \text{(ms.)} \\ + .321(t - 5.5)^2 & t > 5.5 \\ + .096(t - 6.2)^2 & t > 6.2 \\ + .044(t - 8.1)^2 & t > 8.1 \end{cases}$$

(v) Position of the flame front (cm)

$$x = 45 - 3t + .357 \begin{cases} - .073t^2 & t > 0 \\ - .110(t - 2.9)^2 & t > 2.9 \quad \text{(ms.)} \\ + .357(t - 4.2)^2 & t > 4.2 \\ - .015(t - 5.3)^2 & t > 5.3 \\ + .051(t - 6.2)^2 & t > 6.2 \end{cases}$$

These are the formulas used in drawing the graphs in Fig. 9 and the burning region in Fig. 8.

APPENDIX IV

On Analytical and on Numerical Methods for Direct Treatment of the One-dimensionalized Thermo-hydrodynamical Non-Linear Equations

As was mentioned in previous sections, analytical rather than purely numerical solutions of the equations are desirable in order to evaluate simply the relative importance of the various physical parameters involved in pulse-jets. Unfortunately, explicit analytical solutions of the non-linear equations are obtainable at present only in rather special cases. For any explicit numerical case it is possible to apply numerical difference equation (or graphical) methods, even if empirical curves are involved. Whatever the method it apparently must involve considerations of the propagation of discontinuity fronts or 'characteristics' such as are indicated below.

For the one-dimensionalized pulse-jet the thermo-hydrodynamical equations (for a not necessarily perfect gas) take the following forms in terms of Eulerian coordinates (x,t) and dependent variables p, τ, u (cf. p. 10,11):

1. $\theta = H(p,\tau)$ (equation of state) (θ = temperature).
2. $\dot{E} - pD\tau = DU$ (equation of internal energy U) ($D = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ in Eulerians, or $D = \frac{\partial}{\partial t}$ in pseudo-Lagrangians (m,t)). If the internal energy U (per unit mass) is expressed in terms of p and τ then $DU = U_p Dp + U_\tau D\tau$.
3. $\tau_t + u\tau_x - \tau u_x - \tau u S'(x)/S(x) = 0$ (equation of continuity).
4. $u_t + uu_x + \tau p_x = 0$ (equation of motion).

5. $F = u + f$ (equation of flame speed F relative to the jet tube, in terms of the 'intrinsic flame speed' f relative to the gas.

In the above equations $\dot{E}$ and f depend on local turbulence and other parameters which require experimental study. Equation 1. depends on the composition of the gas. If there are no shocks p and u are continuous, and if there are also no temperature discontinuities (such as occur at the interface between fresh and previously burned mixtures) then τ is continuous. Derivatives of such continuous functions may involve discontinuities across certain curves ('characteristics') whose differential equations in terms of dx and dt may be found as illustrated below:

In the customary notation let

6. $H]$ = jump in the value of a function H across a discontimuity. Then if p, u, τ are continuous across a curve whose direction in the (x,t) plane is determined by the ratio of dt to dx:

7. $p_x]dx + p_t]dt = dp] = 0$, $u_x]dx + u_t]dt = du] = 0$,

$\tau_x]dx + \tau_t]dt = d\tau] = 0$ (assuming that $p_x, p_t, u_x, u_t, \tau_x$, and τ_t are continuous on each side of the discontinuity curve.)

Furthermore by 2.-4.

8.
$$\begin{cases} \{\tau_t] + \tau_x]u\}\,(p + U_\tau) + \{p_t] + p_x]u\}\,U_p = \dot{E}] \\ \{\tau_t] + \tau_x]u\} - u_x]\tau = S'(x)]\,\tau u/S(x) \\ \{u_t] + u_x]u\} + p_x]\tau = 0 \end{cases}$$

The relations 8. may be used to eliminate any three of the six jumps $p_t], p_x], \tau_t], \tau_x], u_t], u_x]$ in 7. and thus to provide three linear equations for the remaining three jumps. The three equations are homogeneous if $\dot{E}]$ and $S'(x)]$ are both zero, which is true if $dx \neq Fdt$ and $dx \neq 0$ (i.e. if the characteristic is not a flame boundary or is not a fixed x where the cross-section has a discontinuity in its slope). If the equations are homogeneous the determinant of the system must vanish for at least one of the jumps to be non-zero. This determinant relation yields the following characteristics (in differential form).

9. $dx = u\,dt$ (i.e. $dm = 0$ in pseudo-Lagrangian coordinates),

10. $dx = (u \pm c)dt$ with $c = \tau\sqrt{(p+U_\tau)/U_p}$ the local sound speed relative to the medium.

For 9. the relations 7. and 8., with $\dot{E}] = S'(x)] = 0$, make all the jumps zero except possibly $\tau_x]$ and $\tau_t]$ (but $\tau_t] + \tau_x]u = 0$).

For 10. the relations 7. and 8. with $\dot{E}] = S'(x)] = 0$ determine all the jumps in terms of any (non-zero) one of them.

The possibility of $\dot{E}] \neq 0$ or of $S'(x) \neq 0$ is covered by the following:

The case $u \pm c = 0$ for which 10. reduces to $dx = 0$, corresponds to sonic flow which apparently is not attained in standard pulse-jets. The case $u \pm c = F$ for which 10. reduces to $dx = Fdt$, corresponds to a sonic flame speed which also apparently is not normally attained in pulse-jets. The case $u = 0 \neq F$ for which 9. becomes $dx = 0$, introduces no contradictions even if $S'(x)] \neq 0$. The remaining special case $u = F$ for the characteristic 9. is not exceptional unless $\dot{E}] \neq 0$ at the flame boundary (with intrinsic flame speed $f = 0$). In this last case continuity of τ must be abandoned (cf. p. 22) and the last equation in 7. must be deleted and all functions of τ in 8. must be included behind the jump signs before using the equations to determine the jumps in terms of a prescribed $\dot{E}]$ at the interface concerned.

Having determined the differential equations of the only curves (characteristics) along which discontinuities may 'propagate' there remains the problem of determining how p, τ and u vary after starting with prescribed values (cf. p. 26).

I. *Some explicit analytical solutions.* For an isentropic gas with $\dot{E} = 0$ 3. determines p as an 'adiabatic' function of τ (with $c = \tau\sqrt{-p_\tau}$), and 9. ceases to represent a true characteristic since then all the jumps across it must be zero. Moreover(14,(15 along the characteristics

$dx = (u \pm c)dt$ the quantities $u \pm \sigma$ respectively (with $\sigma = -\int \frac{c d\tau}{\tau}$) are constant if the tube is straight (S(x)const.). If the adiabatic law in this case takes the form $p\tau^{\gamma} = \text{const}$, where $\gamma = (2n + 3)/(2n + 1)$ (with integer $n \geqq 0$), then the general solution may be written in the following form:

$$\text{11.} \quad \begin{cases} x - ut = \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^{n}[G(u + \sigma) + H(u - \sigma)] \\ -ct = \sigma\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^{n}\left[\frac{G(u+\sigma)+H(u-\sigma)}{\sigma}\right] \end{cases}$$

where $G(u+\sigma)$ and $H(u-\sigma)$ are 'arbitrary' functions which are determined by the initial and boundary conditions, and which have continuous (n+1)th order derivatives except for values of their arguments corresponding to characteristics on which actual discontinuities arise. Explicit determinations of G and H are possible if the initial and boundary conditions are not too involved. The method appears to be most practical if piecewise linear functions are used in the conditions and it may be useful for example in the discussion of the non-linear equations in the region above E_o in Fig. 8, page 25. Details will not be considered here.

II. *General numerical solutions.* By replacing differentials by differences (of sufficiently small size), rather accurate numerical solutions of the thermo-hydrodynamical equations 1.-5. subject to prescribed initial and boundary conditions may be found. The type of step-by-step calculation required is indicated by the following:

Suppose that p, τ and u are prescribed at suitably closely spaced points along a curve in an (x,t) diagram, with boundary conditions given at the ends. (Normally the basic curve is the 'initial curve' $t = 0$ from which subsequent curve points are found as indicated below.) To determine points (and the associated values of p, τ and u) on the next curve in the step-by-step process, the relations 2.-4. and the differentials

12. $dp = p_x dx + p_t dt$, $d\tau = \tau_x dx + \tau_t dt$, $du = u_x dx + u_t dt$,

may be used thus:

Let (x,t) and $(x + \delta x, t + \delta t)$ be two adjacent points on the former curve and let $(x + \Delta x , t + \Delta t)$ be a nearby point. In order to keep track of discontinuities and boundaries it is apparently best to choose the third point to lie at the intersection of a discontinuity line (a characteristic, a boundary or a discontinuity line for $S'(x)$) through the first point and a discontinuity line (of a different family) through the second point. It is then possible to calculate the values of p, τ and u at the third point in terms of the values at the first two, using differences in place of differentials in 12. ($dx \sim \Delta x$ or δx, and $dt \sim \Delta t$ or δt). The changes Δp, $\Delta\tau$ and Δu corresponding to Δx, Δt and $\delta x, \delta t$, satisfy linear equations which can be solved explicitly. Details of the method are somewhat lengthy but are not difficult and so will not be presented here.

REFERENCES

(0) See R.A.E. (Farnborough, England) reports, especially No. 2-1317 S: "Pressure measurements on the F.Z.G. 76 flying bomb motor" by Wolfe and Luck (1944).

(1) See Wright Field Power Plant memorandum report T SEPL-5-673-55: "Preliminary Testing of German Robot Bomb Engine" by Bogert and Worth (18 Nov., 1944).

(2) Aircraft Engine Research Lab. N.A.C.A. (Cleveland) Mar. 16, 1945 Memorandum "Performance Tests of Ford MX-544 Intermittent Jet".

(3) See in particular BuAer (U.S.N. Engineering Experiment Station, Annapolis, Md.) Jet Propulsion Note No. 20: "J.B.-2 flame motion data taken from Wright Field high speed motion picture." By Edelman (1 June, 1944).

(4) See Aerojet progress reports.

(5) General Electric Data Folder No. 4T 278: "The intermittent jet cycle" by N. P. Bailey (Sept. 1944). Also: Jet Propelled Missiles Panel (OSRD for Coordinator of Research and Development, U. S. Navy Department) Report No. 92 by N. P. Bailey and H. A. Wilson.

(6) See eg. P. M. Morse's "Vibration and Sound" p. 201 (McGraw Hill 1936), and Rayleigh's "Theory of Sound" §311 and Appendix A (Dover 1945).

(7) Applied Mathematics Group, NY University, Memorandum No. 73: "Motion of gas in half-open pipes" by H. Lewy (Oct. 17, 1944).

(8) See eg. W. Jost's Explosions-und Verbrennungs-vorgange in Gasen" Chapter III § 6 (Edwards Bros. 1943). Recent work on flames and turbulence by Chambre and Lin is expected to appear in the Journal of Aeronautical Science.

(9) See eg. N. P. Bailey's "Heat Engineering" p. 13 Wiley (1942).

(10) See eg. Lamb's "Hydrodynamics" Chap. I (Dover 1945).

(11) See NDRC Applied Math. Panel Report 38.2R: "Supersonic Flow and Shock Waves" by Courant and Friedrichs (Aug. 1944).

(12) See reference (6) for "open end correction".

(13) See eg. Dodge and Thompson's "Fluid Mechanics" Ch. IX (especially p. 213) (McGraw Hill 1937).

(14) See eg. The linearized forms in Love and Pidduck's paper "On Lagrange's Ballistic Problem" T.R.S. London A222, p. 167-226 (1922), and Riemann's fundamental paper in his 'Geo. math. Werke', p. 145 (Leipzig 1876).

(15) See eg. P. de Haller, "The application of a graphic method to some dynamic problems in gases", pp. 6-24 Sulzer Technical Review No. 1 (1945) (Swiss). Three German reports, received just as the present AMG-NYU article was being printed, use analogous numerical and graphical methods (for isentropic flow without true combustion)*. They are:
Schultz-Grunow ZWB/FB/RE-2015 - 1 and 2 "Gasdynamische Untersuchungen am Verpuffungsstrahlrohr". (1943)
Sauer ZWB/FB/RE/1864 - 1 and 2 "Zur Theorie des Strahlrohrs". (1943)

(16) For a general discussion of the perturbation series method, see Frank-Mises, "Die Differential und Integralgleichungen der Mechanik und Physik, II", p. 127.

---

*Analogous numerical methods for cases involving flames can be developed and would be of value for explicit numerical examples. Cf. Appendix IV.

NYU
AMG
151

c.1

MacDonald
A gas dynamical formulation for waves & combustion in pulse-jets.

Date Due

5/[illegible]/59
9/25/59
11/8/61
3/1/62
9/16/62
12/3/63 ILL
Univ of N Dakota
ILL @ UI -10/16/69
ILL Eng [illegible] 11/11/70
ILL Eng Lib 1/14/71
ILL Purdue due 5/6/78
ILL/4.1.84
~~ILL/1.19.86~~

PRINTED IN U. S. A.

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.